ANALYTIC FUNCTIONS
WITH RECTIFIABLE RADIAL IMAGES
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ABSTRACT. We give a simple sufficient condition for an analytic function in
the unit disk to have a radial image of finite length.

1. Introduction. If \( f \) is an analytic function defined on the unit disk \( D \equiv \{z \in \mathbb{C} : |z| < 1\} \), we define the \textit{radial variation function of} \( f \),
\[
V(f, \cdot) : [0, 2\pi) \to [0, \infty) \cup \{\infty\},
\]
by the rule
\[
V(f, \theta) \equiv \int_{(0,1)} |f'(re^{i\theta})| \, dr, \quad \text{all } \theta \in [0, 2\pi).
\]
In the present paper we prove that \( V(f, \theta) \) is finite for at least one \( \theta \in [0, 2\pi) \) if \( f \) is of moderate growth in the unit disk and \( f' \) is bounded on some arc tending to
the boundary of \( D \).

\begin{theorem}
Suppose \( f \) is analytic in \( D \), \( \mu \in [0,1) \) and
\[
\sup_{z \in D} |f(z)|(1 - |z|)^\mu < \infty.
\]
In addition, suppose \( \gamma : [0,1) \to D \) is continuous and one-to-one, \( \lim_{t \uparrow 1} |\gamma(t)| = 1 \), and
\[
(1) \quad \sup_{t \in [0,1)} |f'(\gamma(t))| < \infty.
\]
Then there exists \( \theta \in [0, 2\pi) \) such that \( V(f, \theta) < \infty \).
\end{theorem}

Theorem 1 is proved in \S 3, by means of the harmonic majorization discussed in
\S 2. In \S 4 we present a corollary directed to an open question concerning the radial
images of bounded functions in the disk.

2. Majorization. The following lemma shows how the \textit{radial} growth of a
function which is subharmonic in the upper half plane is \textit{restricted} in the case that
it is \textit{bounded} on a curve terminating at the origin.
LEMMA 1. Suppose \( h, \Gamma, \) and \( \varepsilon \) meet the following conditions:

1. \( h \) is subharmonic in \( H^+ = \{ z \in \mathbb{C}: \text{Im} \, z > 0 \} \) and \( \varepsilon \in (0, \infty) \).
2. \( h(x + iy) \leq \log \frac{1}{y} \) if \( x + iy \in H^+ \) and \( x^2 + y^2 \leq 1 \).
3. \( \Gamma: [0, 1) \to H^+ \) is continuous and one-to-one, \( \Gamma(0) = i \), \( \lim_{t \to 1} \Gamma(t) = 0 \), and \( |\Gamma(t)| \leq 1 \) for all \( t \in [0, 1) \).
4. \( \sup_{t \in (0,1)} h(\Gamma(t)) < \infty \).

Then we have

\[
\sup_{y \in (0,1]} [h(iy) + \left( \frac{1}{2} + \varepsilon \right) \log y] < \infty.
\]

PROOF. Choose \( M \in (0, \infty), a \in (\pi/2, \pi) \) so that \( \pi/2a < 1/2 + \varepsilon \), \( h(T(t)) < M \) for all \( t \in [0,1) \), and \( h(e^{i\theta}) \leq M \) for all \( \theta \in [\pi - \alpha, \alpha] \). Finally, select \( \delta \in (0, \infty) \).

We shall prove

\[
h(iy_0) \leq M + 2\delta + \frac{\pi}{2\alpha} \log \frac{1}{\sin \alpha} + \frac{\pi}{2\alpha} \log \frac{1}{y_0},
\]

all \( y_0 \in (0,1] \).

Note that (6) implies (5), since \( \pi/2\alpha < 1/2 + \varepsilon \), and both \( 1/\alpha \) and \( \delta \) are arbitrary.

We now prove (6). Corresponding to fixed \( y_0 \) and \( \delta \), choose numbers \( \rho \) and \( d \) such that

\[
0 < d < \rho < y_0, \quad \frac{2\rho y_0}{d^2 + y_0^2} \leq \delta,
\]

Also, introduce the following sets which depend on the chosen number \( d \):

\[
\Gamma_1 = \{ de^{i\theta}: 0 \leq \theta \leq \pi \} \cup \{ re^{i\alpha}: d \leq r \leq 1 \} \cup \{ e^{i\theta}: \pi/2 \leq \theta \leq \alpha \},
\]

\[
\Gamma_2 = \{ de^{i\theta}: 0 \leq \theta \leq \pi \} \cup \{ re^{i(\pi - \alpha)}: d \leq r \leq 1 \} \cup \{ e^{i\theta}: \pi - \alpha \leq \theta \leq \pi/2 \},
\]

\[
t^* = \inf\{ t \in [0,1) : |\Gamma(t)| = d \},
\]

\[
\gamma^* = \{ \Gamma(t): 0 \leq t \leq t^* \},
\]

\[
\mathcal{L} = \{ z \in \mathbb{C}: \Gamma_1 \cup \gamma^* \text{ separates } z \text{ from } \infty \},
\]

\[
\mathcal{R} = \{ z \in \mathbb{C}: \Gamma_2 \cup \gamma^* \text{ separates } z \text{ from } \infty \}.
\]

We claim \( iy_0 \in \mathcal{L} \cup \mathcal{R} \cup \gamma^* \). This follows from Janisewski’s Theorem [1, p. 362], since the set

\[
(\Gamma_1 \cup \gamma^*) \cap (\Gamma_2 \cup \gamma^*) = \{ de^{i\theta}: 0 \leq \theta \leq \pi \} \cup \gamma^*
\]

is connected and \( \Gamma_1 \cup \Gamma_2 \) separates \( iy_0 \) from \( \infty \).

We may therefore assume \( iy_0 \in \mathcal{L} \); indeed, (6) is immediate if \( iy_0 \in \gamma^* \), and if \( iy_0 \in \mathcal{R} \) we merely replace \( h \) and \( \Gamma \) by their reflections in the imaginary axis.

Assuming that \( iy_0 \in \mathcal{L} \), let \( \Omega \) denote the set of complex numbers which may be connected to \( iy_0 \) by a path not crossing \( \Gamma_1 \cup \gamma^* \). By definition of \( \Omega \), it follows that \( \text{Bdry} \ \Omega \subset \Gamma_1 \cup \gamma^* \) (here \( \text{Bdry} \ \Omega \) denotes the \( \mathbb{C} \)-boundary of \( \Omega \)). Moreover, if we set \( m = \min\{|d \cdot \sin(\alpha)|, \inf_{z \in \Gamma_1 \cup \gamma^*} \text{Im} \, z| \}, \) we have

\[
\bar{\Omega} \subset S \equiv \{ z \in \mathbb{C}: d \leq |z| \leq 1 \text{ and } \text{Im} \, z \geq m \}.
\]

Indeed, since \( m > 0 \), any point outside \( S \) either lies on \( \Gamma_1 \) or may be connected to \( \infty \) without crossing \( \Gamma_1 \cup \gamma^* \). So, if \( z \in \Omega \) and \( z \notin S \), we could connect \( iy_0 \) to
z and z to ∞ without crossing $\Gamma_1 \cup \gamma^*$: but this contradicts the assumption that $iy_0 \in \mathcal{L}$. Therefore, $\Omega$ is a compact subset of $H^+$, $\overline{\Omega} \subset S$, and

$$\text{Bdry } \Omega \subset (\Gamma_1 \cup \gamma^*) - \{d, -d\}.$$ 

We may now complete the proof. For $z = x + iy \in H^+$ define

$$u(z) \equiv (M + \delta) + \left( \log \frac{|z|}{\sin \alpha} \right) \left( \frac{\arg z}{\alpha} \right) + \rho \left[ \frac{y}{(x+d)^2+y^2} + \frac{y}{(x-d)^2+y^2} \right].$$

Since $(\log |z|)(\arg z) = \frac{1}{2} \text{Im}(\log z)^2$ in $H^+$, we see that $u$ is harmonic in $H^+$. Note that each term of $u$ is nonnegative on $\text{Bdry } \Omega$ since $|z| < 1$ for all $z \in \overline{\Omega} \subset S$. Now, if $x^2 + y^2 = d^2$ and $y > 0$, we have

$$u(x + iy) > \frac{\rho}{y} \log \frac{1}{y} \geq h(x + iy),$$

by (7) and our hypotheses. If $x + iy = re^{i\theta}$ and $d \leq r \leq 1$, then

$$u(x + iy) \geq \left( \log \frac{1}{r \sin \alpha} \right) \left( \frac{\theta}{\alpha} \right) = \log \frac{1}{y} \geq h(x + iy).$$

If $z \in \gamma^*$ or $z = e^{i\theta}$ with $\pi/2 \leq \theta \leq \alpha$, then $u(z) > M \geq h(z)$. Since $\text{Bdry } \Omega \subset \Gamma_1 \cup \gamma^* - \{d, -d\}$, we have shown

$$h(z) - u(z) \leq 0 \quad \text{for all } z \in \text{Bdry } \Omega.$$ 

Since $h - u$ is subharmonic in $H^+$ and $\overline{\Omega}$ is a compact subset of $H^+$, we conclude

$$h(z) - u(z) \leq 0 \quad \text{for all } z \in \Omega.$$ 

Since $iy_0 \in \Omega$ we have

$$h(iy_0) \leq u(iy_0) = (M + \delta) + \left( \log \frac{1}{y_0} + \log \frac{1}{\sin \alpha} \right) \frac{\pi}{2\alpha} + \rho \left[ \frac{2y_0}{d^2 + y_0^2} \right].$$

Now (6) follows from the setup in (7).

3. Proof of Theorem 1. From the general hypotheses it follows that

$$\sup_{z \in D} |f'(z)|(|1 - |z|)^{\mu+1} < \infty.$$ 

Now consider hypothesis (1). By theorems of G. MacLane [2, Theorem 1 and Theorem 14 in G. Piranian’s review], there is a dense set of points on the unit circle which are endpoints of asymptotic paths of $f'$ (possibly corresponding to infinite limits). This fact allows us to assume that

$$\lim_{t \uparrow I} \gamma(t) \text{ exists.}$$

For otherwise there is a nonempty open interval $I \subset (0, 2\pi)$ such that for each $\theta \in I$ there is a corresponding sequence $(t_n)$ from $[0, 1)$ with

$$\lim_{n \to \infty} \gamma(t_n) = e^{i\theta}. \quad \text{By the existence of asymptotic paths of } f', \text{ we may choose } \theta_0 \text{ and } \theta_1 \in I, \text{ and a Jordan arc } \sigma : (0, 1) \to D \text{ such that}$$

$$\lim_{t \uparrow 0} \sigma(t) = e^{i\theta_0}, \quad \lim_{t \uparrow 1} \sigma(t) = e^{i\theta_1},$$

$$\lim_{t \uparrow 0} f'(\sigma(t)) = a, \quad \lim_{t \uparrow 1} f'(\sigma(t)) = b,$$
where $a$ and $b$ are elements of the extended complex plane. By elementary topology we see that for each $\varepsilon \in (0, 1)$ we have
\[
\{ \gamma(t) : 1 - \varepsilon < t < 1 \} \cap \{ \sigma(t) : 0 < t < 1 \} \neq \emptyset.
\]
Since $\lim_{t \uparrow 1} |\gamma(t)| = 1$ and $\sup_{t \in [0, 1]} |f'(\gamma(t))| < \infty$, we may conclude that both $a$ and $b$ are (finite) complex numbers. Hence, if $\lim_{t \uparrow 1} \gamma(t)$ does not exist we may replace $\gamma$ by the parametric arc
\[
t \to \sigma \left( \frac{1 + t}{2} \right), \quad t \in [0, 1).
\]
We may therefore assume, without loss of generality, that $\lim_{t \uparrow 1} \gamma(t) = 1$, $\gamma(0) = 0$, and $\Re \gamma(t) \geq 0$, all $t \in [0, 1)$.

Since $\sup_{z \in D} |f'(z)|(1 - |z|)^{\mu + 1} < \infty$ it is easy to show that
\[
\sup_{z \in H^+ \setminus \{0\}} \left| f' \left( \frac{i - z}{i + z} \right) \cdot \frac{2i}{(i + z)^2} \right| (\Im z)^{\mu + 1} < \infty.
\]
Hence, without loss of generality (multiply $f$ by a positive constant), we assume
\[
\left| f' \left( \frac{i - z}{i + z} \right) \cdot \frac{2i}{(i + z)^2} \right| < \left( \frac{1}{y} \right)^{\mu + 1} \quad \text{if } z = x + iy \in H^+ \text{ and } |z| \leq 1.
\]
Define now
\[
\Gamma(t) \equiv \frac{i - \gamma(t)}{1 + \gamma(t)}, \quad t \in [0, 1),
\]
\[
h(z) \equiv \frac{1}{\mu + 1} \log \left| f' \left( \frac{i - z}{i + z} \right) \cdot \frac{2i}{(i + z)^2} \right|, \quad z \in H^+.
\]
Then $h, \Gamma$ meet the conditions of Lemma 1 because of our simplifying assumptions. If $\varepsilon \in (0, \infty)$ we conclude
\[
\sup_{y \in (0, 1)} \left[ \frac{1}{\mu + 1} \log \left| f' \left( \frac{1 - y}{1 + y} \right) \cdot \frac{2}{(1 + y)^2} \right| + (\frac{1}{2} + \varepsilon) \log y \right] < \infty
\]
and hence that
\[
\sup_{y \in (0, 1)} \left| f' \left( \frac{1 - y}{1 + y} \right) \right| \frac{2y^{(1/2 + \varepsilon)(1 + \mu)}}{(1 + y)^2} < \infty.
\]
Since $\mu \in [0, 1)$ we may choose $\varepsilon$ so that $(\frac{1}{2} + \varepsilon)(\mu + 1) < 1$ and obtain
\[
\int_{[0, 1]} |f'(r)| dr = \int_{(0, 1]} \left| f' \left( \frac{1 - y}{1 + y} \right) \right| \cdot \frac{2}{(1 + y)^2} dy < \infty.
\]
This establishes Theorem 1.

4. Corollary for bounded functions. Although the question of the rectifiability of radial images originates in Rudin's paper [3], it is still unknown whether all the radial images of a bounded analytic function $f$ can be of infinite length. However, by Corollary 1 (stated below), this is impossible if $f'$ has less than the maximal density of zeros possible for the derivative of a bounded analytic function in the disk.
To see this, define $A(f') \equiv \{ a \in D : f'(a) = 0 \}$ whenever $f$ is an analytic function in the disk. Then, if $f$ is bounded, we have
\[
\sum_{a \in A(f')} (1 - |a|)^{1+\varepsilon} < \infty, \quad \text{for all } \varepsilon > 0
\]
(see [4, pp. 204-205]). By Corollary 1, if $V(f, \theta) = \infty$ for all $\theta \in [0, 2\pi)$ it must actually be the case that $\sum_{a \in A(f')} (1 - |a|) = \infty$.

**COROLLARY 1.** Suppose $f$ is analytic in $D$, $\sum_{a \in A(f')} (1 - |a|) < \infty$, $\mu \in [0, 1)$, and $\sup_{z \in D} |f(z)|(1-|z|)^\mu < \infty$. Then there exists $\theta \in [0, \pi)$ such that $V(f, \theta) < \infty$.

**PROOF.** Let $B$ denote the Blaschke product with zero set $A$ and set $g \equiv f'/B$. Since $\log |g|$ is harmonic in $D$ and $D$ is simply connected, there is a simple parametric arc $\gamma : [0, 1) \to D$ such that $\gamma(0) = 0$, $\lim_{t \uparrow 1} |\gamma(t)| = 1$, and $\log |g(\gamma(t))| = \log |g(0)|$ for all $t \in [0, 1)$. Thus $|f'(\gamma(t))| \leq |g(0)|$ for all $t$. Now we apply Theorem 1.

**REFERENCES**


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