PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

BANACH SPACE PROPERTIES OF CIESIELSKI-POL'S $C(K)$ SPACE

G. GODEFROY, J. PELANT, J. H. M. WHITFIELD AND V. ZIZLER

(Communicated by William J. Davis)

ABSTRACT. A $C(K)$ space X_0 which Ciesielski and Pol show does not continuously linearly inject into any $c_0(\Gamma)$ has an equivalent C^∞-norm, is Lipschitz equivalent to a $c_0(\Gamma)$, and the density character of X_0 is equal to the w^*-density character of X_0^*.

1. Introduction. In nonseparable Banach space theory one of the important questions is that of injectability of the space into $c_0(\Gamma)$. For example any weakly compactly generated Banach space or more generally any space analytic in its weak topology continuously linearly injects into $c_0(\Gamma)$ [**2**, **12**, **8**]. In [**9**], Johnson and Lindenstrauss found the first example of a Banach space X with Fréchet differentiable norm that is not a subspace of any weakly compactly generated space. Their space continuously linearly injects into c_0 and the w^*-density character of X^* is $\aleph_0$. Later on, Aharoni and Lindenstrauss proved in [**1**] that the space X of Johnson and Lindenstrauss is Lipschitz equivalent to a $c_0(\Gamma)$ and thus provides a long sought example of two Lipschitz equivalent spaces that are not isomorphic. The space X of Johnson and Lindenstrauss contains a subspace Y which is isometrically isomorphic to c_0 and such that X/Y is isomorphic to $c_0(\Gamma)$. We show that spaces with this property admit equivalent C^∞-norms (Lemma 1).

Recently, Ciesielski and Pol have found an example X_0 of a space similar in structure to that of Johnson-Lindenstrauss space, which moreover has some additional striking properties [**4**]. Their space X_0 is a $C(K)$ space where K is a "ladder system compact" (see definition below). Ciesielski and Pol show that X_0 does not linearly continuously inject into any $c_0(\Gamma)$, while we show in this note that X_0 factors through its subspace which is isomorphic to $c_0(\Gamma_1)$ to a space isomorphic to $c_0(\Gamma_2)$. Following Aharoni and Lindenstrauss in [**1**] we then show that X_0 is Lipschitz equivalent to a $c_0(\Gamma)$. X_0 has nice renorming properties: It admits an equivalent C^∞-norm (Lemma 1), an equivalent locally uniformly rotund norm (since renormability by a locally uniformly rotund norm has the three space property). Since $K^{(3)}$, the third derived set of a compact K in the definition of X_0, is empty, the results of Deville [**6**] imply that X_0 has an equivalent norm $\|\cdot\|$ such that both $\|\cdot\|$ and its dual norm $\|\cdot\|^*$ on X_0^* are locally uniformly rotund. Therefore the space X_0 allows us to sharpen a result of Dashiell and Lindenstrauss in [**5**], where

Received by the editors June 4, 1986 and, in revised form, June 4, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B05, 46B20; Secondary 54C35, 54G20.

Key words and phrases. Renorming, C^∞-norms, three space problem, injections into $c_0(\Gamma)$.

The research of the third author was supported in part by a grant from NSERC, Canada.

The research of the fourth author was supported in part by a University of Alberta grant.

the first example of strictly convex space which does not linearly continuously inject into any $c_0(\Gamma)$ was found. The space of Dashiell and Lindenstrauss cannot have an equivalent locally uniformly rotund norm since it contains an isomorph of l_∞ [**10**, **11**]. The space X_0 of Ciesielski and Pol not only does not linearly continuously inject into any $c_0(\Gamma)$, but has a much stronger property (see the statement of Theorem 1).

The notation in this note is standard in the Banach space theory. The spaces are assumed to be real and differentiability is meant in a continuous Fréchet sense. A Banach space X is said to be Lipschitz equivalent to a Banach space Y if there is a (nonlinear) one-to-one map T of X onto Y such that both T and T^{-1} satisfy the Lipschitz condition. $\operatorname{dens} X$ is the smallest cardinality such that there is a dense set of X of cardinality $\aleph$. w^*-$\operatorname{dens} X^*$ is defined similarly. The set of all natural numbers is denoted by ω and the cardinality of continuum by c.

2. Definition of Ciesielski-Pol's space and the main result. The following definition is taken from [**4**, p. 686].

Let B be a Bernstein set in the real line R (i.e. for every perfect set $P \subset R$, $P \cap B \neq \emptyset$ and $P \cap (R \backslash B) \neq \emptyset$).

Let $\{A_\alpha, \alpha < 2^\omega\}$ be an enumeration of all countable subsets of $R \backslash B$ with uncountable closure. Choose by transfinite induction distinct points $a_\alpha \in \overline{A}_\alpha \cap B$ and for each $\alpha < 2^\omega$, choose and fix a sequence $C_\alpha = \{a_\alpha(i),\ i \in \omega\} \subset A_\alpha$ converging to a_α. Let us give the real line R the following locally compact topology:

Each point of $R \backslash \{a_\alpha\}_\alpha$ is isolated, while a base of neighborhoods of a point a_α are the sets $(a_\alpha \cup C_\alpha) \backslash F$, where F are finite sets in C_α. Finally, let $K = R \cup \infty$ be the one-point compactification of R with the given topology. Then K is called a ladder system compact and the space $C(K)$ of all real valued continuous functions on K endowed with the usual sup-norm will be called the Ciesielski-Pol space X_0.

The following theorem collects known information on X_0 together with the results in this note.

THEOREM 1. *The space X_0 of Ciesielski and Pol defined above admits an equivalent C^∞-norm, is Lipschitz equivalent to a $c_0(\Gamma)$ and* $\operatorname{dens} X_0 = w^*$-$\operatorname{dens} X^* = c$. *$X_0$ also has an equivalent norm $\|\cdot\|$ such that both $\|\cdot\|$ and its dual norm $\|\cdot\|^*$ on X_0^* are locally uniformly convex* [**6**]. *There is a subspace $Y \subset X_0$, Y isomorphic to a $c_0(\Gamma_1)$ such that X_0/Y is isomorphic to a $c_0(\Gamma_2)$. However, from* [**4**], *if (B_1, w) is the unit ball of X_0 endowed with the weak topology, then there is no continuous one-to-one map of (B_1, w) into any $c_0(\Gamma)$ endowed with its weak topology.*

Let us recall that a norm $\|\cdot\|$ on a Banach space X is called locally uniformly rotund if $\lim \|x_n - x\| = 0$ whenever $x_n, x \in X$ are such that $\lim 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$.

3. Proofs of the results.

LEMMA 1. *Let X be a Banach space and Y be a subspace of X such that Y is isomorphic to a space $c_0(\Gamma)$. Assume that X/Y has an equivalent C^k-norm, where $k \in \omega$ or $k = \infty$. Then X admits an equivalent C^k-norm.*

PROOF. We use a technique of Kuiper whose construction of an equivalent C^∞-norm on $c_0(\Gamma)$ is shown in [**3**, p. 896].

First, by a proper extension of a norm given on Y by its isomorphism to $c_0(\Gamma)$ we may suppose that Y is isometric to $c_0(\Gamma)$. Let T be a lifting operator of $Y^* \simeq X^*/Y^\perp \simeq l_1(\Gamma)$ into X^*, $\|T\| \leq 1$ and $\varphi_\alpha = T\delta_\alpha$, $\alpha \in \Gamma$, where δ_α are the unit vectors of $l_1(\Gamma)$.

Let φ be a real-valued C^∞-function on the real line R which is even, $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $|t| \leq 9/8$, $\varphi'(t) < 0$ for $t \in (9/8, 2), \varphi(t) = 0$ for $|t| \geq 2$ and φ is concave on the set $\{t \in R; \varphi(t) \geq 1/2\}$.

Let $\alpha(t)$ be a real-valued C^∞-function on the real line R such that α is even, $\alpha(t) = 0$ for $|t| \leq 1/4$, $\alpha(t) = 1$ for $|t| \geq 1/2$, $\alpha'(t) > 0$ for $t \in (1/4, 1/2)$ and α is convex on $\{t \in R; \alpha(t) \leq 1/2\}$. Let $\psi(\hat{x})$ be a function on X/Y defined by $\psi(\hat{x}) = 1 - \alpha(\|\hat{x}\|)$, where $\hat{x}$ denotes the coset of X/Y given by $x \in X$ and $\|\hat{x}\|$ is an equivalent C^k-norm on X/Y, $\|\cdot\| \geq$ the original quotient norm of X/Y. Then the function ψ on X/Y is a C^k-function on X/Y, $0 \leq \psi \leq 1$, $\psi(\hat{x}) = \psi(\hat{y})$ if $\|\hat{x}\| = \|\hat{y}\|$, $\psi(\hat{x}) = 1$ for $\|\hat{x}\| \leq 1/4$, $\psi(\hat{x}) = 0$ for $\|\hat{x}\| \geq 1/2$, $\psi'(\hat{x})(\hat{x}) < 0$ for $1/4 < \|\hat{x}\| < 1/2$ and ψ is concave on the set $\{\hat{x} \in X/Y; \psi(\hat{x}) \leq 1/2\}$. (Notice that $\psi'(\hat{x})(\hat{x})$ denotes the Fréchet derivative of ψ at a point $\hat{x}$ in the direction $\hat{x}$.) Define a real-valued function Φ on X by

$$\Phi(x) = \psi(\hat{x}) \prod_{\alpha \in \Gamma} \varphi(\varphi_\alpha(x)).$$

Finally, let $|||\cdot|||$ be the Minkowski functional of the set

$$S = \{x \in X; \Phi(x) \geq 1/2\}.$$

We will show that $|||\cdot|||$ is an equivalent C^k-norm on X. Let us first show that Φ is a well-defined C^k-function on X. Let $x \in X$ be such that $\|\hat{x}\| \leq 3/4$. First of all notice that the set $F = \{\alpha \in \Gamma; |\varphi_\alpha(x)| \geq 1\}$ is finite. If not, then for an infinite sequence φ_{α_n}, $n \in \omega$, we would have $|\varphi_{\alpha_n}(x)| \geq 1$, and choosing φ_0 a w^*-limit point of φ_{α_n}'s, we would have $\varphi_0 \in Y^\perp$ (since $\varphi_\alpha = \delta_\alpha$ on $Y = c_0(\Gamma)$). Moreover, since all $\|\varphi_{\alpha_n}\| \leq 1$, $\|\varphi_0\| \leq 1$. On the other hand, $|\varphi_0(x)| \geq 1$ and thus $\|\hat{x}\| \geq 1$, a contradiction with the fact that $\|x\| \leq 3/4$. To show that Φ is a well-defined C^k-function on X, suppose first that $x \in X$ is such that $\|\hat{x}\| \leq 1/2$. Then it follows from the above argument and from equicontinuity of φ_α's that there is a neighborhood $U(x)$ of x and a finite set $F \subset \Gamma$ such that if $y \in U(x)$ and $\alpha \notin F$, then $\varphi(\varphi_\alpha(y)) = 1$. If $x \in X$ is such that $\|\hat{x}\| \geq 1/2$, then there is a neighborhood $U(x)$ of x such that if $y \in U(x)$, then $\|\hat{y}\| > 1/2$ and then $\psi(\hat{y}) = 0$.

From these arguments and from the differentiability properties of φ and ψ it follows that Φ is well defined and is a C^k-function.

The origin is an interior point of S. On the other hand, if $x \in S$, then $\|\hat{x}\| \leq 1/2$ and $|\varphi_\alpha(x)| \leq 2$ for all $\alpha \in \Gamma$. Therefore, if $f \in X^*$ is of norm ≤ 1 and $f|Y$ denotes its restriction to Y, then $T(f|Y) - f \in Y^\perp$ and has norm ≥ 2 (T is a lifting operator considered above in this proof). Therefore, since $\|\hat{x}\| \leq 1/2$, it follows that $|(T(f|Y) - f)(x)| \leq 1$. Since $|T\delta_\alpha(x)| = |\varphi_\alpha(x)| \leq 2$ for all $\alpha \in \Gamma$ and since the unit ball of $l_1(\Gamma)$ is the closed hull of δ_α's, it follows that $|T(f|Y)(x)| \leq 2$. Therefore $|f(x)| = |f(x) - T(f|Y)(x)| + |T(f|Y)(x)| \leq 1 + 2 = 3$. This is true for every $x \in S$ and every $f \in X^*$, $\|f\| \leq 1$. Hence S is bounded.

Clearly, S is a symmetric set in X. We will now show that S is a convex set. To see this suppose that for $x, y \in X$, $\Phi(x) \geq 1/2$, $\Phi(y) \geq 1/2$ and $t \in [0,1]$. We will show that $\Phi(tx + (1-t)y) \geq 1/2$. First of all it is an elementary fact that if x_i and y_i are positive real numbers and $t \in [0,1]$, then if $\prod x_i \geq 1/2$ and $\prod y_i \geq 1/2$, then $\prod(tx_i + (1-t)y_i) \geq 1/2$. Therefore if $x, y \in S$ and $t \in [0,1]$, then

$$(*) \qquad (t\psi(\hat{x}) + (1-t)\psi(\hat{y})) \prod_{\alpha\in\Gamma} t\varphi(\varphi_\alpha(x)) + (1-t)\varphi(\varphi_\alpha(y)) \geq \frac{1}{2}.$$

Since $0 \leq \varphi \leq 1$ and $0 \leq \psi \leq 1$, $\Phi(x) \geq 1/2$ and $\Phi(y) \geq 1/2$ imply that

$$\psi(\hat{x}) \geq 1/2, \quad \psi(\hat{y}) \geq 1/2, \quad \varphi(\varphi_\alpha(x)) \geq 1/2 \quad \text{and} \quad \varphi(\varphi_\alpha(y)) \geq 1/2.$$

Because of concavity of φ and ψ on the desired regions, it follows from $(*)$ that

$$\begin{aligned} &\psi(t\hat{x} + (1-t)\hat{y}) \prod_{\alpha\in\Gamma} \varphi(\varphi_\alpha(tx + (1-t)y)) \\ &\quad = \psi(t\hat{x} + (1-t)\hat{y}) \prod_{\alpha\in\Gamma} \varphi(t\varphi_\alpha(x) + (1-t)\varphi_\alpha(y)) \\ &\quad \geq (t\psi(\hat{x}) + (1-t)\psi(\hat{y})) \prod_{\alpha\in\Gamma} t\varphi(\varphi_\alpha(x) + (1-t)\varphi(\varphi_\alpha(y))) \\ &\quad \geq 1/2. \end{aligned}$$

Therefore S is a convex set.

Finally, since $|||x||| = \lambda > 0$ such that $\Phi(\lambda x) = 1/2$, the Implicit Function Theorem guarantees that $|||\cdot|||$ is a C^k-norm (away from the origin).

We omit the lengthy but straightforward check of this fact but notice that the denominator in the formula involved is not zero (at least one term in the product has its derivative by λ negative).

LEMMA 2. *If for a compact $K, K^{(\omega)}$, the ωth derived set of K, is empty, then $C(K)$ has an equivalent C^∞-norm.*

PROOF It is enough to show, by induction on $n \in \omega$, the following statement V_n: For every compact K such that $K^{(n)} = \varnothing$, $C(K)$ has an equivalent C^∞-norm.

If $K^{(1)} = \varnothing$, then K is finite and $C(K)$ has an equivalent C^∞-norm.

Suppose now V_n is true and assume that K is a compact such that $K^{(n+1)} = \varnothing$.

Consider the set $E = \{f \in C(K); f|K^{(1)} \equiv 0\}$. E is a subspace of $c_0(K)$ and as such has an equivalent C^∞-norm (Kuiper [**3**, p. 896 or Lemma 1]).

Let Q be the restriction map of $C(K)$ to $C(K^{(1)})$. The Tietze extension theorem guarantees that Q is a quotient map with the kernel E. Thus $C(K)/E$ is isomorphic to $C(K^{(1)})$. Since $(K^{(1)})^{(n)} = \varnothing$, it follows from the induction hypothesis that $C(K^{(1)})$ has an equivalent C^∞-norm. Lemma 1 can now be used to see that $C(K)$ has an equivalent C^∞-norm.

Following Aharoni and Lindenstrauss in [**1**] we get

LEMMA 3. *Let X_1 be the subspace of the Ciesielski-Pol space X_0 consisting of all $f \in X_0$ such that $f(\infty) = 0$. Then X_1 is Lipschitz equivalent to a $c_0(\Gamma)$.*

PROOF (An adaptation of that in [**1**]). Let K be the ladder system compact in the definition of X_0. Then $K^{(1)} = \{a_\alpha\}_\alpha \cup \infty$.

Let $E = \{f \in X_1; f|K^{(1)} \equiv 0\}$. Let Q denote the restriction map of X_1 to $K^{(1)}$. Then Tietze extension theorem guarantees that Q maps X_1 onto a subspace X_2 of $c_0(K^{(1)})$ formed by functions which are 0 at ∞ and that X_1/E is isomorphic to X_2. X_1/E is isomorphic to X_2.

We now use the idea of Aharoni and Lindenstrauss in [**1**] to find a lifting ψ of Q, from X_2 to X_1. Let y be a function from X_2 and

$$y = \sum_{n=1}^{t} a_n e_{\alpha_n} - \sum_{m=1}^{s} b_m e_{\alpha'_m}$$

with $a_1 \geq a_2 \geq a_3 \geq \cdots$, $b_1 \geq b_2 \geq b_3 \geq \cdots$, $0 \leq s,t \leq \infty$ be the unique representation of y as a difference of disjointly supported positive elements y^+ and y^- (assume that $\alpha_i \neq \alpha_j$ and $\alpha'_i \neq \alpha'_j$ for $i \neq j$) where $e_{\alpha_n}, e_{\alpha'_n}$ are the unit vectors in $c_0(K^{(1)}), \alpha_n, \alpha'_n \neq \infty$. Put

$$M_n = (\alpha_n \cup C_{\alpha_n}) \backslash \left(\bigcup_{i=1}^{n-1} C_{a_{\alpha_i}} \right),$$

$$M'_m = (\alpha'_m \cup C_{\alpha'_m}) \backslash \left(\bigcup_{j=1}^{m-1} a_{\alpha'_j} \right),$$

where $C_{a_{\alpha_i}}$ are the ladders for a_{α_i} in the definition of Ciesielski-Pol's space X_0.

Furthermore, put for $y \in X_2$

$$\psi(y) = \sum_{n=1}^{t} a_n \chi_{M_n} - \sum_{m=1}^{s} b_m \chi_{M'_m},$$

where χ_{M_n} ($\chi_{M'_m}$) denotes the characteristic function of M_n (M'_m) in K. $\psi(y)$ is an extension of y on K.

Now use the fact from [**1**] that if $y \in X_2$ and $k \in K \backslash \{a_\alpha\}_\alpha$, then

$$\psi(y)(k) = \operatorname{dist}(y^+, Z_k) - \operatorname{dist}(y^-, Z_k),$$

where $Z_k = \operatorname{sp}\{e_\alpha, k \notin C_\alpha\}$.

Therefore ψ is a Lipschitz lifting of Q and thus the Bartle-Graves map $Tu = (u - \psi(Qu), Qu)$ can be used to see that X_1 is Lipschitz equivalent to $X_1 \oplus E$.

Finally observe that E is isometric to $c_0(K \backslash K^{(1)})$ (use the restriction map) and X_2 is isometric to $c_0(\{a_\alpha\}_\alpha)$. Therefore X_1 is Lipschitz equivalent to $c_0(K \backslash K^{(1)}) \oplus c_0(\{a_\alpha\})$ and thus to a $c_0(\Gamma)$.

PROOF OF THEOREM 1. First of all observe that the space X_1 from Lemma 3 is isomorphic to the Ciesielski-Pol space X_0. To see this fact consider a fixed element a_α in the definition of X_0 above and let $a_\alpha(1)$ be the first element of the ladder C_α corresponding to a_α. The hyperplane $Z = \{f \in X_0, f(a_\alpha(1)) = 0\}$ is isomorphic to the space X_0, since it is clearly isomorphic to the space $C(K \backslash \{a_\alpha(1)\})$ (use the restriction map) and the space $C(K \backslash \{a_\alpha(1)\})$ is in turn isomorphic to $C(K)$ (use the shift on the ladder C_α and identity outside C_α to construct a homemorphism of K and $K \backslash \{a_\alpha(1)\}$). Since any two hyperplanes Z_1, Z_2 (through the origin) of a given Banach space X are always isomorphic (both of them are isomorphic to $(Z_1 \cap Z_2) \oplus R$), X_1 is isomorphic to X_0.

Since for the ladder system compact K in the definition of Ciesielski-Pol's space X_0, $K^{(3)} = \varnothing$, the statement of renormability of X_0 by C^∞-norm directly follows from Lemma 1.

The statement on factorization of X_0 is verified at the end of the proof of Lemma 3.

To show the statement about density characters, observe first that there are $\mathbf{c}$ disjoint perfect sets in R and since each of them intersects $K\backslash K^{(1)}$, it follows that cardinality of $K\backslash K^{(1)}$ is equal to $\mathbf{c}$. Furthermore, since X_1 is Lipschitz equivalent to $c_0(K\backslash K^{(1)}) \oplus c_0(\{a_\alpha\})$ it follows that $\operatorname{dens} X_0 = \operatorname{dens} X_1 = \mathbf{c}$. On the other hand, if $E = \{f \in C(K); f|K^{(1)} \equiv 0\}$, then

$$w^*\text{-dens}\, X_0^* = w^*\text{-dens}\, X_1^* \geq w^*\text{-dens}\, E^* = w^*(c_0(K\backslash K^{(1)})) = c$$

(cf. [**10**, Proposition 2.2]).

The statement that there is no continuous injection of (B_1, w) into any $(c_0(\Gamma), \omega)$ was proved in [**4**, Proof of Lemma 5.2].

REMARKS AND OPEN PROBLEMS. 1. Assuming Generalized Continuum Hypothesis, Ciesielski and Pol show in [**4**] that a similarly constructed space $X_0' = C(K)$ where K is a ladder system compact of ordinals, is Lindelöf in its weak topology yet having no weakly continuous injection into any $c_0(\Gamma)$ in its weak topology. One easily checks that most statements of this note remain valid for X_0'.

2. Due to the nonexistence of a weak-weak continuous injection of X_1 into any $c_0(\Gamma)$, the quotient space X_2 of X_1 has no weak-weak continuous lifting for a quotient map of X_1 onto X_2.

3. It would be interesting to decide if X_0 has an equivalent uniformly Gâteaux differentiable norm and if X_0 admits C^∞-partitions of unity.

ACKNOWLEDGMENT. The work of this paper was completed while the first and last named authors were visiting the University of Missouri at Columbia. The authors thank this University for its hospitality.

REFERENCES

1. I. Aharoni and Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), 281–283.
2. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. **88** (1968), 35–46.
3. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, Indiana Univ. Math. J. **15** (1966), 877–898.
4. K. Ciesielski and R. Pol, *A weakly Lindelöf function space $C(K)$ without any continuous injection into $c_0(T)$*, Bull. Polish Acad. Sci. Math. **32** (1984), 681–688.
5. F. K. Dashiell and J. Lindenstrauss, *Some examples concerning strictly convex norms on $C(K)$ spaces*, Israel J. Math. **16** (1973), 329–342.
6. R. Deville, *Problèmes de renormages*, J. Funct. Anal. **68** (1986), 117–129.
7. G. Godefroy, S. Troyanski, J. Whitfield and V. Zizler, *Three-space problem for locally uniformly rotund renormings of Banach spaces*, Proc. Amer. Math. Soc. **94** (1985), 647–652.
8. S. P. Gulko, *On the structure of the spaces of continuous functions and on the hereditary paracompactness of these spaces*, Uspekhi Mat. Nauk **34** (1979), 33–40. (Russian)
9. W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974), 219–230.
10. J. Lindenstrauss, *Weakly compact sets, their topological properties and the Banach spaces they generate* (R. D. Anderson, ed.), Ann. of Math. Studies, no. 69, Princeton Univ. Press, Princeton, N. J., 1972, pp. 235–274.

11. S. Troyanski, *Equivalent norms and minimal systems in Banach spaces*, Studia Math. **43** (1972), 125–138.
12. L. Vasak, *On the generalization of weakly compactly generated Banach spaces*, Studia Math. **70** (1979), 11–19.

EQUIPE D'ANALYSE, UNIVERSITÉ PARIS VI, PLACE JUSSIEU, F-75230 PARIS, FRANCE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211

MATHEMATICAL INSTITUTE, CZECHOSLOVAK ACADEMY OF SCIENCES, ZITNA 25, PRAGUE, CZECHOSLOVAKIA

DEPARTMENT OF MATHEMATICAL SCIENCES, LAKEHEAD UNIVERSITY, THUNDER BAY, ONTARIO, CANADA P7B 5E1

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1