STARLIKENESS AND CONVEXITY FROM INTEGRAL MEANS OF THE DERIVATIVE

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ABSTRACT. If \( f \) is analytic in \( |z| < 1 \) and normalized: \( f(0) = f'(0) - 1 = 0 \), then \( f \) is univalent and starlike in \( |z| < I(f) \), where

\[
I(f) = \sup_r \left\{ (\frac{2\pi}{2})^{-1} \int_0^{2\pi} |f'(re^{it})|^2 \, dt \right\}^{-1/2}, \quad 0 \leq r < 1.
\]

Furthermore, there exists a normalized \( f \) such that \( I(f) < 1 \) and that \( f' \) vanishes at a point on \( |z| = I(f) \).

If \( f \) is analytic and normalized in \( |z| < 1 \), then \( f \) is univalent and convex in \( |z| < I(f)/2 \).

1. Introduction. Let \( F \) be the family of functions \( f \) analytic in \( D = \{ |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \). The radius of starlikeness \( \sigma(f) \) of \( f \in F \) is the largest \( r \) such that \( f \) is univalent in \( D(r) = \{ |z| < r \} \) and \( cf(z) \in f(D(r)) \) for all \( z \in D(r) \) and all \( c, 0 < c < 1 \), where \( 0 < r \leq 1 \). Setting

\[
M_2(r, f') = \left\{ (\frac{2\pi}{2})^{-1} \int_0^{2\pi} |f'(re^{it})|^2 \, dt \right\}^{1/2}, \quad 0 \leq r < 1,
\]

and

\[
\Phi_2(f) = \sup_{0 \leq r < 1} \frac{r}{M_2(r, f')} \quad \text{for} \quad f \in F,
\]

we begin with

THEOREM 1. \( \sigma(f) \geq \Phi_2(f) \) for \( f \in F \).

Set \( \|f'\|_2 = \lim_{r \to 1} M_2(r, f') \leq +\infty \). Since

\[
\Phi_2(f) \geq \|f'\|_2^{-1} \geq 0,
\]

it follows that

\[
\sigma(f) \geq \|f'\|_2^{-1} \quad \text{for} \quad f \in F,
\]

a known result [G1, Theorem 23, p. 187] (see also [Gd2, II, p. 95]).

However, the estimate (2) is of no value in case \( \|f'\|_2 = +\infty \), while Theorem 1 remains available because \( \Phi_2(f) > 0 \) for each \( f \in F \).

We can construct \( f \in F \) such that

\[
\sigma(f) = \Phi_2(f) > \|f'\|_2^{-1} > 0.
\]

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2. **Proof of Theorem 1.** See [D1] for general references for the mean $M_p(r, h)$ and the norm $\|h\|_p$ of $h$ analytic in $D$, $0 < p < +\infty$, and $0 < r < 1$; thus, $H^p = \{h; \|h\|_p < +\infty\}$, the Hardy class.

For $f \in F$ we set

$$\Phi_p(r, f) = r\{1 + M_p(r, f' - 1)^p\}^{-1/p}, \quad 0 \leq r < 1,$$

and

$$\Phi_p(f) = \sup_{0 \leq r < 1} \Phi_p(r, f), \quad 0 < p < +\infty;$$

we note that $\Phi_2(r, f) = r/M_2(r, f')$, so that $\Phi_2(f)$ is the same as in Theorem 1. Apparently, $\Phi_p(f) \geq (1 + \|f' - 1\|^p)^{-1/p}$. Theorem 1 is now the case $p = 2$ in

**THEOREM 2.** $\sigma(f) \geq \Phi_p(f)$ for $f \in F$ ($1 \leq p \leq 2$).

Comments on Theorem 2 for $1 \leq p < 2$ will be given in Remark 1. For the proof of Theorem 2 we shall make use of two lemmas.

**LEMMA 1.** If $h(z) = \sum_{n=1}^{\infty} b_n z^n \in F$, and if $\sum_{n=2}^{\infty} n|b_n| \leq 1$, then $\sigma(h) = 1$.

See [Gd1, Theorem 1; CK, Theorem 3; D2, p. 73 and Gd2, I, p. 128].

**LEMMA 2 [D1, Theorem 6.1, p. 94].** If $h(z) = \sum_{n=0}^{\infty} b_n z^n \in H^p$ ($1 \leq p \leq 2$), then

$$\left( \sum_{n=0}^{\infty} |b_n|^q \right)^{1/q} \leq \|h\|_p \quad (1/p + 1/q = 1),$$

where the left-hand side is $\sup_{n \geq 0} |b_n|$ if $p = 1$.

To prove Theorem 2 we may suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \neq z.$$

For each fixed $r$, $0 < r < 1$, we set $R = \Phi_p(r, f)$. Then, $0 < R < r$, and for $h(z) = f'(rz) - 1$, Lemma 2 yields that

$$\left\{ \sum_{n=2}^{\infty} n|a_n|r^{n-1}q \right\}^{1/q} \leq \|h\|_p = M_p(r, f' - 1).$$

The Hölder inequality enables us to have

$$\sum_{n=2}^{\infty} n|a_n|R^{n-1} = \sum_{n=2}^{\infty} n|a_n|r^{n-1}(R/r)^{n-1} \leq M_p(r, f' - 1) \left\{ \sum_{n=2}^{\infty} (R/r)^{pn-p} \right\}^{1/p} = 1.$$

With the aid of Lemma 1 we obtain $\sigma(g) = 1$ for $g(z) = R^{-1}f(Rz) \in F$, whence $\sigma(f) \geq R$. Since $r$ is arbitrary this completes the proof of Theorem 2.

To construct $f \in F$ with (3) we fix $r$, $0 < r < 1$, and then we choose $A$ such that $r^{-1} < A < 2^{-1}(r + r^{-3})$. Then, $f$ is defined by

$$f(z) = A^2 z - (A^2 - 1)Ar \log\{rA/(rA - z)\};$$
this is analytic in \( \{|z| < rA\} \). Since

\[
f'(z) = (-A) \cdot \frac{z/r - 1/A}{1 - z/(rA)},
\]
it follows that \( M_2(r, f') = A \), and hence \( \Phi_2(f) \geq r/A \). On the other hand, \( \sigma(f) \leq r/A \) because \( f'(r/A) = 0 \). We thus arrive at \( \sigma(f) = \Phi_2(f) = r/A \). For the norm of \( f' \in H^2 \) we have

\[
\|f'\|_2^2 \geq A^2 (r^{-1} - A^{-1})^2 (2\pi)^{-1} \int_0^{2\pi} |1 - (rA)^{-1} e^{it}|^{-2} dt
\]
\[
= A^2 (r^{-1} - A^{-1})^2 \{1 - (rA)^{-2}\}^{-1} > (A/r)^2.
\]

Therefore, \( f \) satisfies (3).

**REMARK 1.** The existence of \( f \in F \) with \( \sigma(f) = \Phi_p(f) > (1 + \|f' - 1\|_p)^{-1/p} > 0 \) \((1 < p < 2)\) is unknown. Also, it seems not easy to compare \( \Phi_p(r, f) \) with \( \Phi_2(r, f) \) in case \( p < 2 \). We observe this for \( p = 1 \): sometimes, \( \Phi_1(r, f) < \Phi_2(r, f) \), and sometimes, \( \Phi_1(r, f) > \Phi_2(r, f) \).

Given \( r, 0 < r < 1 \), we observe that \( f(z) = z + 2^{-1}r^{-1}z^2 \in F \) satisfies \( \Phi_1(r, f) < \Phi_2(r, f) \) because

\[
1 + M_1(r, f' - 1) = 2 > 2^{1/2} = M_2(r, f').
\]

On the other hand, let \( 0 < r < 1 \). Then, the function \( f(z) = rG(r^{-1}z) \), where

\[
G(z) = z + 2^{-1}Az^2 - 3^{-1}Az^3, \quad A > 4\pi/(\pi^2 - 8),
\]

satisfies \( \Phi_1(r, f) > \Phi_2(r, f) \). Actually,

\[
1 + M_1(r, f' - 1) = 1 + \|G' - 1\|_1 = 4\pi - 4^2 A + 1
\]
\[
< (1 + 2A^2)^{1/2} = \|G'\|_2 = M_2(r, f'),
\]

where we make use of \( \int_0^{2\pi} |1 - e^{it}| dt = 8 \).

3. **Radius of convexity.** The radius of convexity \( \kappa(f) \) of \( f \in F \) is the largest \( r \) \((0 < r \leq 1)\) such that \( f \) is univalent in \( D(r) \) and \( cf(z) + (1 - c)f(w) \in f(D(r)) \) for all \( z, w \in D(r) \) and all \( c, 0 < c < 1 \).

**THEOREM 1C.** \( \kappa(f) \geq \Phi_2(f)/2 \) for \( f \in F \).

We have no information on the sharpness. Theorem 1C is actually the case \( p = 2 \) in

**THEOREM 2C.** \( \kappa(f) \geq \Phi_p(f)/2 \) for \( f \in F \) \((1 \leq p \leq 2)\).

We follow the same lines as in the proof of Theorem 2, where, in this case, Lemma 1 is replaced by

**LEMMA 1C.** If \( h(z) = \sum_{n=1}^\infty b_nz^n \in F \), and if \( \sum_{n=2}^\infty n|b_n| \leq 1 \), then \( \kappa(h) \geq 1/2 \).

The estimate is exact since \( \kappa(h_0) = 1/2 \) for \( h_0(z) = z - 2^{-1}z^2 \). The proof of Lemma 1C depends on the following lemma.
Lemma 3. (See, for example, [Gd1, Theorem 1].) If \( h(z) = \sum_{n=1}^{\infty} b_n z^n \in F \), and if \( \sum_{n=2}^{\infty} n^2 |b_n| \leq 1 \), then \( \kappa(h) = 1 \).

Lemma 1C is an exercise in [D2, p. 73], and the proof is in a few lines which we shall give for completeness. For \( g(z) = 2h(z/2) = \sum_{n=1}^{\infty} c_n z^n \in F \) we have

\[
\sum_{n=2}^{\infty} n^2 |c_n| \leq \sum_{n=2}^{\infty} n |b_n| \leq 1
\]

by \( n^2 - n + 1 \leq 1 \) (\( n \geq 2 \)), so that \( \kappa(g) = 1 \) by Lemma 3, and hence \( \kappa(h) \geq 1/2 \).

Remark 2. Since \( \Phi_2(f_n) = (1 + n^2)^{-1/2} \to 0 \) as \( n \to \infty \) for \( f_n(z) = z + 2^{-1} n z^2 \in F \), it follows that \( \Phi_2(F) = 0 \), where

\[
\Phi_2(F_1) = \inf \{ \Phi_2(f) ; f \in F_1 \} \quad \text{for} \quad F_1 \subset F.
\]

For what subfamily \( F_1 \) of \( F \) have we \( \Phi_2(F_1) > 0 \)? A typical example is the family \( S \) of all \( f \in F \) univalent in \( D \). We have

\[
\Phi_2(S) \geq \sup_{0 < x < 1} \left\{ x / \phi(x) \right\}^{1/2} \equiv c = 0.164 \ldots,
\]

where

\[
\phi(x) = (1 - x)^{-5} (x^3 + 11x^2 + 11x + 1).
\]

Remembering the known constants [D2, pp. 44 and 98, Gd2, I, pp. 119 and 121], due to H. Grunsky and R. Nevanlinna:

\[
c_G = \inf \{ \sigma(f) ; f \in S \} = \tanh(\pi/4) = 0.6557 \ldots,
\]

\[
c_N = \inf \{ \kappa(f) ; f \in S \} = 2 - \sqrt{3} < c_G/2,
\]

we have by Theorem 1C the estimates

\[
(4) \quad c \leq \Phi_2(S) \leq 2c_N < c_G.
\]

It would be interesting to fill the considerable gap between \( c \) and \( 2c_N = 0.535 \ldots \).

For the proof of (4) we make use of the de Branges theorem [B] that \( |a_n| \leq n \) (\( n \geq 2 \)) for \( f(z) = \sum_{n=1}^{\infty} a_n z^n \in S \). Setting \( x = r^2 \) for \( 0 < r < 1 \) we obtain

\[
M_2(r, f')^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \leq \sum_{n=1}^{\infty} n^4 x^{n-1} = \phi(x),
\]

so that

\[
\Phi_2(f) \geq c = \{ x_0 / \phi(x_0) \}^{1/2} \quad \text{for} \quad x_0 = 0.84 \ldots.
\]

Remark 3. Theorem 1C also follows directly from Lemma 3. For \( f(z) = \sum_{n=1}^{\infty} a_n z^n \neq z \) and for \( 0 < r < 1 \), let \( Q = Q(r, f) \) be the real root of the equation

\[
A((1 + x)/(1 - x)^3 - 1) = 1, \quad A = M_2(r, f')^2 - 1.
\]

We shall soon observe that \( 0 < Q < 1 \). Set

\[
\Psi(r, f) = rQ(r, f)^{1/2}, \quad 0 < r < 1.
\]

Then, we can show that

\[
(7) \quad \kappa(f) \geq \sup_{0 < r < 1} \Psi(r, f),
\]
and furthermore,

(8) \[ \Psi(r, f) \geq \Phi_2(r, f)/2 \quad (0 < r < 1), \]

whence Theorem 1C follows.

We consider the cubic curve \( Y = X^3 + BX - 2B \) in the XY-plane, where \( B = A/(1 + A) \). The curve cuts the X-axis at the only one point

\[ X_0 = B^{1/3} \left[ \left( 1 + \left( 1 + \frac{B}{27} \right)^{1/2} \right)^{1/3} + \left( 1 - \left( 1 + \frac{B}{27} \right)^{1/2} \right)^{1/3} \right], \]

and has the straight line

\[ Y = (B + 3)X - 2B - 2 \]

as the tangent at \( X = 1 \); the tangent cuts the X-axis at \( X_1 = 2(B + 1)/(B + 3) \). By an elementary analysis we have \( 0 < X_0 < X_1 < 1 \). Now the solution \( Q \) of (6) is given by \( Q = 1 - X_0 \), so that simple calculations show that

\[ 4^{-1}M_2(r, f')^{-2} \leq 1 - X_1 < Q < 1. \]

We thus have (8).

For the proof of (7) we first note that \( 0 < R = \Psi(r, f) < r \). By the Schwarz inequality we obtain

\[
\left( \sum_{n=2}^{\infty} n^2 |a_n| R^{n-1} \right)^2 = \left( \sum_{n=2}^{\infty} n^2 |a_n| r^{n-1} (R/r)^{n-1} \right)^2 \\
\leq M_2(r, f' - 1)^2 \left( \sum_{n=2}^{\infty} n^2 (R/r)^{2n-2} \right) = 1
\]

by (6), so that \( \kappa(g) = 1 \) for \( g(z) = R^{-1}f(Rz) \) by Lemma 3. We thus have \( \kappa(f) \geq R \), and this completes the proof of (7).

ADDED IN PROOF TO REMARK 2. For the Koebe function \( k \in S \) we have \( c = \Phi_2(k) \geq \Phi_2(S) \). Therefore \( \Phi_2(S) = c \).

REFERENCES


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