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STARLIKENESS AND CONVEXITY FROM INTEGRAL MEANS OF THE DERIVATIVE

SHINJI YAMASHITA

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ABSTRACT. If f is analytic in $|z| < 1$ and normalized: $f(0) = f'(0) - 1 = 0$, then f is univalent and starlike in $|z| < I(f)$, where

$$I(f) = \sup r \left\{ (2\pi)^{-1} \int_0^{2\pi} |f'(re^{it})|^2 \, dt \right\}^{-1/2}, \qquad 0 \leq r < 1.$$

Furthermore, there exists a normalized f such that $I(f) < 1$ and that f' vanishes at a point on $|z| = I(f)$.

If f is analytic and normalized in $|z| < 1$, then f is univalent and convex in $|z| < I(f)/2$.

1. Introduction. Let F be the family of functions f analytic in $D = \{|z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. The radius of starlikeness $\sigma(f)$ of $f \in F$ is the largest r such that f is univalent in $D(r) = \{|z| < r\}$ and $cf(z) \in f(D(r))$ for all $z \in D(r)$ and all c, $0 < c < 1$, where $0 < r \leq 1$. Setting

$$M_2(r, f') = \left\{ (2\pi)^{-1} \int_0^{2\pi} |f'(re^{it})|^2 \, dt \right\}^{1/2}, \qquad 0 \leq r < 1,$$

and

$$\Phi_2(f) = \sup_{0 \leq r < 1} r/M_2(r, f') \quad \text{for } f \in F,$$

we begin with

THEOREM 1. *$\sigma(f) \geq \Phi_2(f)$ for $f \in F$.*

Set $\|f'\|_2 = \lim_{r \to 1} M_2(r, f') \leq +\infty$. Since

$$\Phi_2(f) \geq \|f'\|_2^{-1} \geq 0, \tag{1}$$

it follows that

$$\sigma(f) \geq \|f'\|_2^{-1} \quad \text{for } f \in F, \tag{2}$$

a known result [**G1**, Theorem 23, p. 187] (see also [**Gd2**, II, p. 95]).

However, the estimate (2) is of no value in case $\|f'\|_2 = +\infty$, while Theorem 1 remains available because $\Phi_2(f) > 0$ for each $f \in F$.

We can construct $f \in F$ such that

$$\sigma(f) = \Phi_2(f) > \|f'\|_2^{-1} > 0. \tag{3}$$

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2. Proof of Theorem 1. See [**D1**] for general references for the mean $M_p(r, h)$ and the norm $\|h\|_p$ of h analytic in D, $0 < p < +\infty$, and $0 \le r < 1$; thus, $H^p = \{h; \|h\|_p < +\infty\}$, the Hardy class.

For $f \in F$ we set

$$\Phi_p(r, f) = r\{1 + M_p(r, f' - 1)^p\}^{-1/p}, \qquad 0 \le r < 1,$$

and

$$\Phi_p(f) = \sup_{0 \le r < 1} \Phi_p(r, f), \qquad 0 < p < +\infty;$$

we note that $\Phi_2(r, f) = r/M_2(r, f')$, so that $\Phi_2(f)$ is the same as in Theorem 1. Apparently, $\Phi_p(f) \ge (1 + \|f' - 1\|_p^p)^{-1/p}$. Theorem 1 is now the case $p = 2$ in

THEOREM 2. *$\sigma(f) \ge \Phi_p(f)$ for $f \in F$ $(1 \le p \le 2)$.*

Comments on Theorem 2 for $1 \le p < 2$ will be given in Remark 1. For the proof of Theorem 2 we shall make use of two lemmas.

LEMMA 1. *If $h(z) = \sum_{n=1}^{\infty} b_n z^n \in F$, and if $\sum_{n=2}^{\infty} n|b_n| \le 1$, then $\sigma(h) = 1$.*

See [**Gd1**, Theorem 1; **CK**, Theorem 3; **D2**, p. 73 and **Gd2**, I, p. 128].

LEMMA 2 [**D1**, Theorem 6.1, p. 94]. *If $h(z) = \sum_{n=0}^{\infty} b_n z^n \in H^p$ $(1 \le p \le 2)$, then*

$$\left(\sum_{n=0}^{\infty} |b_n|^q\right)^{1/q} \le \|h\|_p \qquad (1/p + 1/q = 1),$$

where the left-hand side is $\sup_{n \ge 0} |b_n|$ if $p = 1$.

To prove Theorem 2 we may suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \not\equiv z.$$

For each fixed r, $0 < r < 1$, we set $R = \Phi_p(r, f)$. Then, $0 < R < r$, and for $h(z) = f'(rz) - 1$, Lemma 2 yields that

$$\left\{\sum_{n=2}^{\infty} (n|a_n|r^{n-1})^q\right\}^{1/q} \le \|h\|_p = M_p(r, f' - 1).$$

The Hölder inequality enables us to have

$$\begin{aligned}\sum_{n=2}^{\infty} n|a_n|R^{n-1} &= \sum_{n=2}^{\infty} n|a_n|r^{n-1}(R/r)^{n-1} \\ &\le M_p(r, f' - 1)\left\{\sum_{n=2}^{\infty} (R/r)^{pn-p}\right\}^{1/p} = 1.\end{aligned}$$

With the aid of Lemma 1 we obtain $\sigma(g) = 1$ for $g(z) = R^{-1} f(Rz) \in F$, whence $\sigma(f) \ge R$. Since r is arbitrary this completes the proof of Theorem 2.

To construct $f \in F$ with (3) we fix r, $0 < r < 1$, and then we choose A such that $r^{-1} < A < 2^{-1}(r + r^{-3})$. Then, f is defined by

$$f(z) = A^2 z - (A^2 - 1)Ar \log\{rA/(rA - z)\};$$

this is analytic in $\{|z| < rA\}$. Since

$$f'(z) = (-A) \cdot \frac{z/r - 1/A}{1 - z/(rA)},$$

it follows that $M_2(r, f') = A$, and hence $\Phi_2(f) \geq r/A$. On the other hand, $\sigma(f) \leq r/A$ because $f'(r/A) = 0$. We thus arrive at $\sigma(f) = \Phi_2(f) = r/A$. For the norm of $f' \in H^2$ we have

$$\begin{aligned}\|f'\|_2^2 &\geq A^2(r^{-1} - A^{-1})^2(2\pi)^{-1} \int_0^{2\pi} |1 - (rA)^{-1}e^{it}|^{-2}\, dt \\ &= A^2(r^{-1} - A^{-1})^2\{1 - (rA)^{-2}\}^{-1} > (A/r)^2.\end{aligned}$$

Therefore, f satisfies (3).

REMARK 1. The existence of $f \in F$ with $\sigma(f) = \Phi_p(f) > (1+\|f'-1\|_p^p)^{-1/p} > 0$ $(1 \leq p < 2)$ is unknown. Also, it seems not easy to compare $\Phi_p(r, f)$ with $\Phi_2(r, f)$ in case $p < 2$. We observe this for $p = 1$: sometimes, $\Phi_1(r, f) < \Phi_2(r, f)$, and sometimes, $\Phi_1(r, f) > \Phi_2(r, f)$.

Given r, $0 < r < 1$, we observe that $f(z) = z + 2^{-1}r^{-1}z^2 \in F$ satisfies $\Phi_1(r, f) < \Phi_2(r, f)$ because

$$1 + M_1(r, f' - 1) = 2 > 2^{1/2} = M_2(r, f').$$

On the other hand, let $0 < r < 1$. Then, the function $f(z) = rG(r^{-1}z)$, where

$$G(z) = z + 2^{-1}Az^2 - 3^{-1}Az^3, \qquad A > 4\pi/(\pi^2 - 8),$$

satisfies $\Phi_1(r, f) > \Phi_2(r, f)$. Actually,

$$\begin{aligned}1 + M_1(r, f' - 1) &= 1 + \|G' - 1\|_1 = 4\pi^{-1}A + 1 \\ &< (1 + 2A^2)^{1/2} = \|G'\|_2 = M_2(r, f'),\end{aligned}$$

where we make use of $\int_0^{2\pi} |1 - e^{it}|\, dt = 8$.

3. Radius of convexity. The radius of convexity $\kappa(f)$ of $f \in F$ is the largest r $(0 < r \leq 1)$ such that f is univalent in $D(r)$ and $cf(z) + (1 - c)f(w) \in f(D(r))$ for all z, $w \in D(r)$ and all c, $0 < c < 1$.

THEOREM 1C. *$\kappa(f) \geq \Phi_2(f)/2$ for $f \in F$.*

We have no information on the sharpness. Theorem 1C is actually the case $p = 2$ in

THEOREM 2C. *$\kappa(f) \geq \Phi_p(f)/2$ for $f \in F$ $(1 \leq p \leq 2)$.*

We follow the same lines as in the proof of Theorem 2, where, in this case, Lemma 1 is replaced by

LEMMA 1C. *If $h(z) = \sum_{n=1}^{\infty} b_n z^n \in F$, and if $\sum_{n=2}^{\infty} n|b_n| \leq 1$, then $\kappa(h) \geq 1/2$.*

The estimate is exact since $\kappa(h_0) = 1/2$ for $h_0(z) = z - 2^{-1}z^2$. The proof of Lemma 1C depends on the following lemma.

LEMMA 3. (See, for example, [**Gd1**, Theorem 1].) If $h(z) = \sum_{n=1}^{\infty} b_n z^n \in F$, and if $\sum_{n=2}^{\infty} n^2 |b_n| \leq 1$, then $\kappa(h) = 1$.

Lemma 1C is an exercise in [**D2**, p. 73], and the proof is in a few lines which we shall give for completeness. For $g(z) = 2h(z/2) = \sum_{n=1}^{\infty} c_n z^n \in F$ we have

$$\sum_{n=2}^{\infty} n^2 |c_n| \leq \sum_{n=2}^{\infty} n|b_n| \leq 1$$

by $n2^{-n+1} \leq 1$ $(n \geq 2)$, so that $\kappa(g) = 1$ by Lemma 3, and hence $\kappa(h) \geq 1/2$.

REMARK 2. Since $\Phi_2(f_n) = (1+n^2)^{-1/2} \to 0$ as $n \to \infty$ for $f_n(z) = z + 2^{-1} n z^2 \in F$, it follows that $\Phi_2(F) = 0$, where

$$\Phi_2(F_1) = \inf\{\Phi_2(f); f \in F_1\} \quad \text{for } F_1 \subset F.$$

For what subfamily F_1 of F have we $\Phi_2(F_1) > 0$? A typical example is the family S of all $f \in F$ univalent in D. We have

$$\Phi_2(S) \geq \sup_{0 \leq x < 1} \{x/\phi(x)\}^{1/2} \equiv c = 0.164\ldots, \tag{4}$$

where

$$\phi(x) = (1-x)^{-5}(x^3 + 11x^2 + 11x + 1).$$

Remembering the known constants [**D2**, pp. 44 and 98, **Gd2**, I, pp. 119 and 121], due to H. Grunsky and R. Nevanlinna:

$$c_G = \inf\{\sigma(f); f \in S\} = \tanh(\pi/4) = 0.6557\ldots,$$

$$c_N = \inf\{\kappa(f); f \in S\} = 2 - \sqrt{3} < c_G/2,$$

we have by Theorem 1C the estimates

$$c \leq \Phi_2(S) \leq 2c_N < c_G. \tag{5}$$

It would be interesting to fill the considerable gap between c and $2c_N = 0.535\ldots$.

For the proof of (4) we make use of the de Branges theorem [**B**] that $|a_n| \leq n$ $(n \geq 2)$ for $f(z) = \sum_{n=1}^{\infty} a_n z^n \in S$. Setting $x = r^2$ for $0 < r < 1$ we obtain

$$M_2(r, f')^2 = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \leq \sum_{n=1}^{\infty} n^4 x^{n-1} = \phi(x),$$

so that

$$\Phi_2(f) \geq c = \{x_0/\phi(x_0)\}^{1/2} \quad \text{for } x_0 = 0.84\ldots.$$

REMARK 3. Theorem 1C also follows directly from Lemma 3. For $f(z) = \sum_{n=1}^{\infty} a_n z^n \not\equiv z$ and for $0 < r < 1$, let $Q = Q(r, f)$ be the real root of the equation

$$A\{(1+x)/(1-x)^3 - 1\} = 1, \qquad A = M_2(r, f')^2 - 1. \tag{6}$$

We shall soon observe that $0 < Q < 1$. Set

$$\Psi(r, f) = rQ(r, f)^{1/2}, \qquad 0 < r < 1.$$

Then, we can show that

$$\kappa(f) \geq \sup_{0<r<1} \Psi(r, f), \tag{7}$$

and furthermore,

$$\Psi(r,f) \geq \Phi_2(r,f)/2 \qquad (0<r<1), \tag{8}$$

whence Theorem 1C follows.

We consider the cubic curve $Y = X^3 + BX - 2B$ in the XY-plane, where $B = A/(1+A)$. The curve cuts the X-axis at the only one point

$$X_0 = B^{1/3}\left[\left\{1+\left(1+\frac{B}{27}\right)^{1/2}\right\}^{1/3} + \left\{1-\left(1+\frac{B}{27}\right)^{1/2}\right\}^{1/3}\right],$$

and has the straight line

$$Y = (B+3)X - 2B - 2$$

as the tangent at $X = 1$; the tangent cuts the X-axis at $X_1 = 2(B+1)/(B+3)$. By an elementary analysis we have $0 < X_0 < X_1 < 1$. Now the solution Q of (6) is given by $Q = 1 - X_0$, so that simple calculations show that

$$4^{-1}M_2(r,f')^{-2} \leq 1 - X_1 < Q < 1.$$

We thus have (8).

For the proof of (7) we first note that $0 < R \equiv \Psi(r,f) < r$. By the Schwarz inequality we obtain

$$\begin{aligned}\left(\sum_{n=2}^{\infty} n^2|a_n|R^{n-1}\right)^2 &= \left(\sum_{n=2}^{\infty} n^2|a_n|r^{n-1}(R/r)^{n-1}\right)^2 \\ &\leq M_2(r,f'-1)^2\left(\sum_{n=2}^{\infty} n^2(R/r)^{2n-2}\right) = 1\end{aligned}$$

by (6), so that $\kappa(g) = 1$ for $g(z) = R^{-1}f(Rz)$ by Lemma 3. We thus have $\kappa(f) \geq R$, and this completes the proof of (7).

ADDED IN PROOF TO REMARK 2. For the Koebe function $k \in S$ we have $c = \Phi_2(k) \geq \Phi_2(S)$. Therefore $\Phi_2(S) = c$.

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKASAWA, SETAGAYA, TOKYO 158, JAPAN