A PROJECTION FORMULA FOR THE ASKEY-WILSON POLYNOMIALS AND AN APPLICATION

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ABSTRACT. A projection formula for \( p_n(x; a, b, c, d|q) \), the Askey-Wilson polynomials, is obtained by using a generalization of Askey and Wilson’s \( q \)-beta integral. The result is used to find a \( q \)-analogue of the Feldheim-Vilenkin formula for ultraspherical polynomials. A \( q \)-analogue of the ultraspherical polynomials, other than the one due to Rogers, is also introduced.

1. Introduction. The main objective of this paper is to find a \( q \)-analogue of the projection formula

\[
(1 - x)^{\alpha + \mu} P_n^{(\alpha + \mu, \beta)}(x) = \frac{2^{\mu} \Gamma(\alpha + \mu + 1)}{\Gamma(\alpha + 1) \Gamma(\mu)} \int_0^1 (1 - y)^\alpha (y - x)^{\mu - 1} P_n^{(\alpha, \beta)}(y) \frac{dy}{(1 + y)^{\alpha n + \alpha + \mu + 1} P_n^{(\alpha, \beta)}(1)},
\]

\(-1 < x < 1, \alpha > -1, \mu > 0\), where \( P_n^{(\alpha, \beta)}(x) \) is the Jacobi polynomial defined by

\[
P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} _2F_1\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right).
\]

A change of variable on both sides of (1.1) gives the accompanying formula

\[
(1 + x)^{\beta + \mu} P_n^{(\beta + \mu, \alpha)}(x) = \frac{2^{\mu} \Gamma(\beta + \mu + 1)}{\Gamma(\beta + 1) \Gamma(\mu)} \int_{-1}^1 (1 + y)^\beta (x - y)^{\mu - 1} P_n^{(\alpha, \beta)}(y) \frac{dy}{(1 - y)^{\beta n + \beta + \mu + 1} P_n^{(\beta, \alpha)}(1)},
\]

\(-1 < x < 1, \beta > -1, \mu > 0\). Askey and Fitch [3] used (1.1) to give a simple proof of Feldheim [10] and Vilenkin’s [23] formula

\[
\frac{C_n^{(\nu)}(\cos \theta)}{C_n^{(1)}} = \frac{2^{\nu + 1/2}}{\Gamma(\nu + 1/2) \Gamma(\nu - \frac{1}{2})} \int_0^{\pi/2} \sin^{2\nu + 2\nu - 2\nu - 1} \phi(1 - \sin^2 \theta \cos^2 \phi)^{n/2} \cdot C_n^{\lambda}(\cos \theta(1 - \sin^2 \theta \cos^2 \phi)^{-1/2}) d\phi, \quad \nu > \lambda > - \frac{1}{2},
\]

where

\[
C_n^{\lambda}(\cos \theta) = \frac{(2\lambda)_n}{(\lambda + 1/2)_n} p_n^{(\lambda - 1/2, \lambda - 1/2)}(\cos \theta)
\]

(1.5)

\[
C_n^{\lambda}(\cos \theta) = \sum_{k=0}^{\frac{n}{2}} \frac{(\lambda)_k (\lambda)_n - k}{k!(n - k)!} \cos(n - 2k)\theta.
\]

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is the ultraspherical polynomial; see Szegö [22]. Feldheim [10] used (1.4) and the positivity of Fejér’s [9] sum $\sum_{k=0}^{n} P_k(x)$ for Legendre polynomials to show that

$$
\sum_{k=0}^{n} \frac{C_k'(\cos \theta)}{C_k'(1)} > 0, \quad 0 \leq \theta < \pi, \quad \nu \geq \frac{1}{2}.
$$

Askey and Fitch [3] used (1.4) to deduce the integral representation

$$
\frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = \int_{-1}^{1} \frac{P_n^{(\beta,\beta)}(y)}{P_n^{(\beta,\beta)}(1)} d\mu(y),
$$

where $\alpha > \beta \geq -1/2$, $-1 \leq x \leq 1$, and $d\mu(y)$ is a positive measure which depends on $x$ but not on $n$.

Askey [2] also gave an elementary proof of Koornwinder’s [12, 13] Laplace-type integral representation for Jacobi polynomials by using (1.1). Laine [15] found an extension of (1.1) to obtain a formula that is equivalent to Koornwinder’s [12, 14] addition formula for Jacobi polynomials.

To find a $q$-analogue of (1.1) we shall need the following extension of the beta-integral

$$
\int_{0}^{\pi} \left\{ (e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos \theta, a_1 a_2 a_3 a_4 a_5) \prod_{j=1}^{5} h(\cos \theta, a_j) \right\} d\theta
$$

$$
= 2\pi \prod_{j=1}^{5} (a_1 a_2 a_3 a_4 a_5 / a_j q^5 / (q q)_\infty \prod_{1 \leq j < k \leq 5} (a_j a_k q)_\infty,
$$

where

$$
(A; q)_\infty = \prod_{j=0}^{\infty} (1 - A q^j), \quad 0 < q < 1,
$$

and $\max(|a_j|) < 1$, $j = 1, \ldots, 5$. If we set $a_5 = 0$ then (1.8) reduces to the $q$-beta integral of Askey and Wilson [6, (2.1)]. Formula (1.8) was given in Rahman [17].

Askey and Wilson [6] introduced the polynomials

$$
p_n(x) = p_n(x; a, b, c, d|q)
$$

$$
= 4\phi_3 \left[ q^{-n}, \quad abcd q^{n-1}, \quad ae^{i\theta}, \quad ae^{-i\theta} \quad ; q, q, q \right], \quad x = \cos \theta,
$$

as a generalization of the Jacobi polynomials and showed that they are orthogonal on $[-1, 1]$ with respect to the weight function

$$
W(x) \equiv W(x; a, b, c, d|q)
$$

$$
= \frac{h(x, 1) h(x, -1) h(x, q^{1/2}) h(x, -q^{1/2})}{h(x, a) h(x, b) h(x, c) h(x, d)} (1 - x^2)^{-1/2}
$$
where $a, b, c, d$ are either real or, if complex, occur in conjugate pairs such that their absolute values are less than 1. We shall assume throughout this paper that $0 < q < 1$. The symbol on the right side of (1.11) represents a basic hypergeometric series defined by

$$
(1.13) \quad r+1\phi_r \left[ \begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n,
$$

where

$$
(A_1, A_2, \ldots, A_m; q)_n = \prod_{j=1}^{m} (A_j; q)_n,
$$

$$(A; q)_n = \begin{cases} 1, & n = 0, \\ (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}), & n = 1, 2, \ldots. \end{cases}
$$

If any one of the parameters $a_1, \ldots, a_{r+1}$ is of the form $q^{-k}$, $k$ a nonnegative integer, then the series in (1.13) terminates, otherwise we require that $|z| < 1$ to ensure convergence. If $z = q$ and $b_1 \cdots b_r = qa_1 \cdots a_{r+1}$, then the $r+1\phi_r$ series is said to be balanced. If $qa_1 \neq a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r$ then the series is called a nearly-poised series of the first kind, and it is a nearly-poised series of the second kind if $qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$. The $r+1\phi_r$ series is called well-poised if $qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$; if, in addition, $a_2 = -a_3 = qa_1^{1/2}$ then we call the series very-well-poised. Note that the $q^3$ series in (1.11) is balanced and terminating.

Since there are four parameters in $p_n(x; a, b, c, d|q)$ there are many ways to choose them so that a continuous $q$-Jacobi polynomial can be defined in the sense that $p_n(x)$ gives $P_n^{(\alpha, \beta)}(x)$ in the limit $q \to 1$. Askey and Wilson [6] took

$$
(1.15) \quad a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\beta+1)/4}, \quad d = -q^{(2\beta+3)/4}
$$
to define

$$
(1.16) \quad P_n^{(\alpha, \beta)}(x|q) = \frac{(q^{(\alpha+1)}; q)_n}{(q; q)_n} 4\phi_3 \left[ \begin{array}{c} q^n, q^{n+\alpha+\beta+1}, q^{(2\alpha+1)/4}e^{i\theta}, q^{(2\alpha+1)/4}e^{-i\theta} \\ q^{\alpha+1}, -q^{(\alpha+\beta+1)/2}, q^{\alpha+1} \end{array} ; q, q \right]
$$
as a $q$-analogue of the Jacobi polynomials. The author [18] chose

$$
(1.17) \quad a = q^{1/2}, \quad b = q^{\alpha+1/2}, \quad c = -q^{\beta+1/2}, \quad d = -q^{1/2}
$$
to introduce

$$
(1.18) \quad P_n^{(\alpha, \beta)}(x|q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q^{1/2}; q^{1/2})_n} 4\phi_3 \left[ \begin{array}{c} q^n, q^{n+\alpha+\beta+1}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta} \\ q^{\alpha+1}, -q^{\beta+1}, q^{1/2} \end{array} ; q, q \right]
$$
as another $q$-analogue. It is easy to see that

$$
(1.19) \quad \lim_{q \to 1} P_n^{(\alpha, \beta)}(x|q) = \lim_{q \to 1} P_n^{(\alpha, \beta)}(x; q) = P_n^{(\alpha, \beta)}(x).
$$

Also, as Askey and Wilson [6] pointed out, these analogues are related by the formula

$$
(1.20) \quad P_n^{(\alpha, \beta)}(x|q^2) = \frac{(-q; q)_n}{(-q^{\alpha+\beta+1}; q)_n} q^{an} P_n^{(\alpha, \beta)}(x; q).
$$
A number of important results for Jacobi polynomials have already been extended by using one or the other of these $q$-extensions (see, for example, Askey and Wilson [6], Gasper and Rahman [11], Rahman [19]), including a projection formula for $P_n^{\alpha,\beta}(x|q)$; see Nassrallah and Rahman [16]. However, we were unable to derive a $q$-analogue of (1.1) by choosing the parameters as in (1.15) or (1.17). So we were forced to consider other possibilities and we found that a projection formula analogous to (1.1) exists if we take

$$a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\alpha+1)/4}, \quad d = -q^{\alpha+(3-2\alpha)/4}.$$  

The basic formula that we shall prove in §2 is

$$\int_{-1}^{1} v(y; a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(abc\mu e^{i\theta}, abc\mu e^{-i\theta}; q)_n}{(abc\mu^2 e^{i\theta}, abc\mu^2 e^{-i\theta}; q)_n} p_n(y; a, b, c, d|q) \, dy$$

$$= g(a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(bc; q)_n}{(bc\mu^2; q)_n} p_n(x; a\mu, b\mu, c\mu, d\mu^{-1}|q)$$

where $x = \cos \theta$, $y = \cos \phi$ and

$$v(y; a_1, a_2, a_3, a_4, a_5|q)$$

$$= \frac{h(y, 1)h(y, -1)h(y, q^{1/2})h(y, -q^{-1/2})h(y, a_1a_2a_3a_4a_5)}{h(y, a_1h(y, a_2)h(y, a_3)h(y, a_4)h(y, a_5)}(1-y^2)^{-1/2},$$

$$g(a_1, a_2, a_3, a_4, a_5|q) = \int_{-1}^{1} v(y; a_1, a_2, a_3, a_4, a_5|q) \, dy$$

$$= 2\pi \prod_{j=1}^{5} \left( \frac{a_1a_2a_3a_4a_5}{a_j}; q \right)_\infty \sum_{1 \leq j \leq 5} \prod_{1 \leq j \leq 5} (a_j a_k; q) \infty,$$

by (1.8). In (1.22) it is assumed that $-1 < a, b, c < 1$ and $0 < \mu < 1$. It is not hard to see that (1.22) is indeed a $q$-analogue of (1.1) if we choose the parameters as given in (1.21).

In §3 we shall derive the following $q$-analogue of the ultraspherical polynomials

$$G_n(\cos \theta; \beta|q)$$

$$= \frac{(-\beta^{1/2} q^{(n+1)/2} e^{-i\theta}, -\beta^{1/2} q^{(1-n)/2} e^{2i\theta}; q)\infty}{(-\beta^{1/2} q^{(n+1)/2} e^{-i\theta}, -\beta^{1/2} q^{(1-n)/2} e^{2i\theta}; q)\infty} e^{2(n/2-[n/2])i\theta}$$

$$\left( \frac{\beta; q}{q; q} \right)_n \frac{3\phi_2}{\left[ \begin{array}{ccc} \beta, & -\beta^{1/2} q^{(1-n)/2} e^{-2i\theta} \\ \beta^{-1} q^{1-n}, & -\beta^{1/2} q^{(1-n)/2} e^{2i\theta} \end{array} \right]}$$

which is quite different and much more complicated than the $q$-ultraspherical polynomials of Rogers (see Askey and Ismail [4]). Formulas (3.5) and (3.7) will show, respectively, that $G_n(\cos \theta; \beta|q)$ is a polynomial when $n$ is even, but not so when $n$ is odd unless it is divided by $e^{i\theta} + \beta^{1/2} e^{-i\theta}$. However, it has the property that

$$\lim_{q \to 1} G_n(\cos \theta; q^\lambda|q) = C_\lambda^n(\cos \theta), \quad \lambda > 0$$

and that there is a $q$-analogue of the Feldheim-Vilenkin formula (1.4) for $G_n(\cos \theta; \beta|q)$, which we will derive in §4.
## 2. Proof of (1.22)

Let \(j, n\) be nonnegative integers with \(j \leq n\) and let \(a, b, \mu, \nu\) be complex numbers with modulus less than 1. Also let \(0 \leq \theta \leq \pi\). Then, by (1.8) and (1.24)

\[
\int_{-1}^{1} v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab\nu|\mu^2 e^{i\phi}, ab\nu|\mu^2 e^{-i\phi}; q)_n} dy
\]

\[
= \int_{-1}^{1} v(y; q^{1-n}/b, \nu|q) = g(aq^j, bq^{n-j}, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q)
\]

\[
= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) q^{(j+1)-2nj} b^{-2j} \frac{(ab, bu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n(q_{1-n}/b, \nu|q)_{1-n}}{(ab\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n(q_{1-n}/b, \nu|q)_{1-n}}
\]

Since, by Watson's formula [7, 8.5 (2)],

\[
\begin{align*}
&= \frac{(b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n}{(ab, b/a; q)_n} \mathcal{W}_7(aq^{1-n}/b; ae^{i\theta}, ae^{-i\theta}, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q, cdq^n) \\
&= \sum_{j=0}^{n} \frac{(aq^{-n}/b, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q)_j (1 - aq^{-2n}/b)}{(q, ac, ad, q/a; q)_j (1 - aq^{-n}/b)} \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab, b/a; q)_n}
\end{align*}
\]

where

\[
\begin{pmatrix}
\begin{aligned}
q^{-n}, \quad abcdq^{n-1}, \\ ab, \\ ac, \\ ad
\end{aligned}
\end{pmatrix}
\]

\[
= \frac{r+3a_{r+2}}{aq^{a_{r+2}}}, q, z
\]

we find that

\[
\begin{align*}
\int_{-1}^{1} v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) p_n(y; a, b, c, d|q) dy \\
&= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(b\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n}{(b/a, ab\nu, b\mu e^{i\theta}, ab\mu e^{-i\theta}; q)_n} \cdot \mathcal{W}_9(aq^{-n}/b; a\mu e^{i\theta}, a\mu e^{-i\theta}, \nu, q^{1-n}/bc, q^{1-n}/bd, q^{-n}/b, q^{-n}/cdq^n, q^{-n}/cdq^n).
\end{align*}
\]

If we choose \(\nu = c\) then the \(\mathcal{W}_9\) series on the right side becomes an \(8\phi_7\) series which, in turn, is expressible as \(p_n(z)\) with \(a, b, c, d\) replaced by \(a\mu, b\mu, c\mu, d\mu^{-1}\), respectively. This completes the proof of (1.22). In order that (1.22) be seen as a \(q\)-analogue of (1.1) we need to impose the additional restrictions mentioned earlier, i.e., \(-1 < a, b, c < 1\) and \(0 < \mu < 1\).
Transforming the $4\phi_3$ series in (1.11) by Sears’ transformation formula [20, (8.3)] and interchanging the parameters we obtain from (1.22) another formula

\[
\int_{-1}^{1} v(y; b, c, d, \mu e^{i\theta}, \mu e^{-i\theta}) (bcd\mu e^{i\theta} \mu c, \mu d e^{-i\theta}; q)_{n} p_n(y; a, b, c, d | q) dy \\
g(b, c, d, \mu e^{i\theta}, \mu e^{-i\theta}) (bc, bd, cd; q)_{n}^{-1} p_n(x; a \mu^{-1}, b \mu, c \mu, d \mu | q)
\]

which can be seen as a $q$-analogue of (1.3) if we specialize the parameters as

\[
a = q^{\alpha + (3 - 2\beta)/4}, \quad b = q^{(2\beta + 1)/4}, \quad c = -q^{(2\beta + 1)/4}, \quad d = -q^{(2\beta + 3)/4}.
\]

3. A $q$-analogue of the ultraspherical polynomials. By (1.11) and [7, 8.5 (2)],

\[
p_n(x; a, aq^{1/2}, -a, -q^{1/2}/a | q) = 4\phi_3 \left[ \begin{array}{c}
q^{-n}, \quad a^2 q^n, \quad a e^{i\theta}, \quad ae^{-i\theta} \\
q^{-n}, \quad a^2 q^n, \quad a e^{i\theta}, \quad ae^{-i\theta}
\end{array} \right; q, q
\]

\[
= (aq^{1/2} - ne^{i\theta}, -aq^{1/2} - ne^{-i\theta}; q)_{n}
\]

\[
8 W_7(-a^2 q^{-n-1/2}; ae^{i\theta}, ae^{-i\theta}, q^{1/2-n}, -q^{-n}, -q^{-n}, q, -a^2 q^{n+1/2}).
\]

Setting $d = -aq^{1/2}$ in Bailey’s formula [21, (3.4.1.16)] we find that

\[
4\phi_3 \left[ \begin{array}{c}
a, \quad b, \quad c, \quad q^{-n} \\
aq/b, \quad aq/c, \quad b^2 c^2 q^{-n-1}/a
\end{array} \right; q, q
\]

\[
= (a^2 q^2/b^2 c^2, -a^1/2 q^{3/2}/bc; q)_{n}
\]

\[
\cdot W_9(-a^3/2 q^{1/2}/bc; a^{1/2}, -a^{1/2}, (aq)^{1/2} - (aq)^{1/2}/b,
\]

\[
- (aq)^{1/2}/c, a^2 q^n + 2/b^2 c^2, q^{-n}/q, q, q
\]

from which, it follows by taking the limit $n \to \infty$ and replacing $a, b, c$ by $q^{-2n}, -q^{1/2-n} e^{i\theta}/a$ and $-q^{1/2-n} e^{-i\theta}/a$, respectively, that

\[
8 W_7(-a^2 q^{-n-1/2}; ae^{i\theta}, ae^{-i\theta}, q^{1/2-n}, -q^{-n}, -q^{-n}, q, -a^2 q^{n+1/2})
\]

\[
= (a^4 q^{2n}, -a^2 q^{1/2-n}; q)_{\infty}
\]

\[
\times 3 \phi_2 \left[ \begin{array}{c}
q^{-2n}, \quad -q^{1/2-n} e^{i\theta}/a, \quad -q^{1/2-n} e^{-i\theta}/a \\
-q^{1/2-n} e^{i\theta}/a, \quad -q^{1/2-n} e^{-i\theta}/a
\end{array} \right; q, a^4 q^{2n}
\]

By Sears’ transformation formula [20, II(a), p. 173],

\[
3\phi_2 \left[ \begin{array}{c}
q^{-2n}, \quad -q^{1/2-n} e^{i\theta}/a, \quad -q^{1/2-n} e^{-i\theta}/a \\
-aq^{1/2-n} e^{i\theta}, \quad -aq^{1/2-n} e^{-i\theta}
\end{array} \right; q, a^4 q^{2n}
\]

\[
= (a^2; q)_{2n} / (-aq^{1/2-n} e^{i\theta}; q)_{2n} 3\phi_2 \left[ \begin{array}{c}
q^{-2n}, \quad a^2, \quad -q^{1/2-n} e^{-i\theta}/a \\
q^{1/2-n}/a^2, \quad -aq^{1/2-n} e^{i\theta}
\end{array} \right; q, q
\]
and hence by combining (3.1), (3.3) and (3.4) we find that

\[ P_n(\cos 2\theta; a, aq^{1/2}, -a, -q^{1/2}/a|q) = \frac{(q; q)_n}{(q^2; q)_n} G_{2n}(\cos \theta; a^2|q), \]

where \( G_n(x; \beta|q) \) is as defined in (1.25). This is a \( q \)-analogue of the quadratic transformation formula [8, 10.9 (21)]:

\[ P_n^{(\lambda-1/2, -1/2)}(\cos 2\theta) = \frac{(1/2)^n}{\lambda^n} C_n^\lambda(\cos \theta). \]

Similarly one can show that

\[ P_n(\cos 2\theta; a, aq^{1/2}, -aq^{1/2}, -q/a|q) = \frac{(q; q)_n}{(a^4; q)_n} \frac{1 + a^2}{(q^{-n}; q)_n e^{i\theta} + ae^{-i\theta}} G_{2n+1}(\cos \theta; a^2|q), \]

which is a \( q \)-analogue of [8, 10.9 (22)]:

\[ \cos \theta P_n^{(\lambda-1/2, 1/2)}(\cos 2\theta) = \frac{(1/2)^{n+1}}{(\lambda)^{n+1}} C_{n+1}^\lambda(\cos \theta). \]

It is clear from (3.5) and (3.7) that \( G_n(x; q^\lambda|q) \) behaves like the ultraspherical polynomials \( C_n^\lambda(x) \) in the limit \( q \to 1 \), but otherwise quite different from and not nearly as nice as Rogers’ continuous \( q \)-ultraspherical polynomials

\[ C_n(\cos \theta; \beta|q) = \sum_{k=0}^{n} \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \cos(n-2k)\theta. \]

The orthogonality and other properties of \( C_n(x; \beta|q) \) are by now well known, mainly through the works of Askey and Ismail [4, 5] and most of these results have proved very useful. We do not expect the functions \( G_n(x; \beta|q) \) to be half as useful. But the fact that they occur in a \( q \)-analogue of an important formula, namely (1.4), for ultraspherical polynomials, is probably reason enough to look at them a bit more closely, study their properties and look for other applications. This we plan to do in a later report.

4. A \( q \)-analogue of the Feldheim-Vilenkin formula. From (1.22) we find that

\[ \int_0^{\pi/2} \rho(\cos 2\phi; a, b, c, \mu e^{2i\phi}, \mu^{-2i\phi}|q) \frac{(abc\mu e^{2i\phi}, abc\mu^{-2i\phi}; q)_n}{(abc\mu^2 e^{2i\phi}, abc\mu^{-2i\phi}; q)_n} \cdot p_n(\cos 2\phi; a, b, c, d|q) \, d\phi \]

\[ = \frac{1}{2} q(a, b, c, \mu e^{2i\phi}, \mu^{-2i\phi}|q) \frac{(bc; q)_n}{(bc\mu^2; q)_n} p_n(\cos 2\theta; a\mu, b\mu, c\mu, d\mu^{-1}|q), \]

where

\[ \rho(\cos 2\phi; a_1, a_2, a_3, a_4, a_5|q) \]

\[ = \frac{h(\cos 2\phi, 1) h(\cos 2\phi, -1) h(\cos 2\phi, q^{1/2}) h(\cos 2\phi, -q^{1/2}) h(\cos 2\phi, a_1 a_2 a_3 a_4 a_5)}{h(\cos 2\phi, a_1) h(\cos 2\phi, a_2) h(\cos 2\phi, a_3) h(\cos 2\phi, a_4) h(\cos 2\phi, a_5)}. \]
Using (3.5) we then have

\[ (4.3) \]

\[
\int_{0}^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -a, e^{2i\theta}, e^{-2i\theta} | q) \frac{(-q^{1/2} a^3 \mu e^{2i\theta}, -q^{1/2} a^3 \mu e^{-2i\theta} | q)_n}{(-q^{1/2} a^3 \mu^2 e^{2i\phi}, -q^{1/2} a^3 \mu^2 e^{-2i\phi} | q)_n} \cdot G_n(\cos \phi; a^2 | q) \, d\phi 
\]

\[
= \frac{1}{2} g(a, aq^{1/2}, -a, e^{2i\theta}, e^{-2i\theta} | q) \frac{(a^4; q)_{2n}}{(a^4 \mu^4; q)_{2n}} G_n(\cos \theta; a^2 | q). 
\]

Similarly, the use of (3.7) gives

\[ (4.4) \]

\[
\int_{0}^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -aq^{1/2}, e^{2i\theta}, e^{-2i\theta} | q) \frac{(-qa^3 \mu e^{2i\theta}, -qa^3 \mu e^{-2i\theta} | q)_n}{(-qa^3 \mu^2 e^{2i\phi}, -qa^3 \mu^2 e^{-2i\phi} | q)_n} \cdot G_{n+1}(\cos \phi; a^2 | q) \, d\phi 
\]

\[
= \frac{1}{2} g(a, aq^{1/2}, -aq^{1/2}, e^{2i\theta}, e^{-2i\theta} | q) \frac{1 + a^2 \mu^2}{1 + a^2} \frac{(a^4; q)_{2n+1}}{(a^4 \mu^4; q)_{2n+1}} \cdot G_{n+1}(\cos \theta; a^2 | q). 
\]

Combining (4.3) and (4.4) we get the formula

\[ (4.5) \]

\[
\int_{0}^{\pi/2} K(\cos \theta, \cos \phi; a^2 | q) \frac{h(\cos 2\phi, -a^3 \mu^2 q^{(n+1)/2})}{h(\cos 2\theta, -a^3 \mu^2 q^{(n+1)/2})} \frac{(\varphi(1-n)/2 + [n/2] e^{2i\phi} | q)_\infty}{\varphi(1-n)/2 + [n/2] e^{2i\phi} | q)_\infty} \cdot \frac{e^{2i(\theta - \phi)}(n/2 - [n/2]) \cdot G_n(\cos \phi; a^2 | q) \, d\phi}{(a^4 \mu^4; q)_{2n}} 
\]

\[
= \frac{(a^4; q)_n}{(a^4 \mu^4; q)_n} G_n(\cos \theta; a^2 | q), 
\]

where

\[ (4.6) \]

\[
K(\cos \theta, \cos \phi; a^2 | q) = \frac{(q, a^4, \mu^2, a^2 \mu^2 | q)_\infty}{\pi(a^4 \mu^4, a^2 | q)_\infty} \frac{h(\cos 4\phi, 1) h(\cos 4\theta, a^2 \mu^2)}{h(\cos 4\phi, a^2) h(\cos 2\phi, e^{2i\theta}) h(\cos 2\phi, e^{-2i\theta})}, \quad |a| < 1, 0 < \mu < 1. 
\]

Replacing \( a, \mu \) by \( q^{\lambda/2}, q^{(\nu - \lambda)/2} \), \( \nu > \lambda > -1/2 \), it can be shown by using the \( q \)-gamma function

\[ (4.7) \]

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^2; q)_\infty} (1 - q)^{1-x}, \quad \Gamma(x) = \lim_{q \to 1} \Gamma_q(x), 
\]

that (4.5) reduces to

\[ (4.8) \]

\[
\frac{C_n^\nu(\cos \theta)}{C_n^\nu(1)} \frac{\sin^{2\nu-1} \theta}{\cos^{n+2\lambda+1} \theta} = \frac{2\Gamma(\nu + 1/2)}{\Gamma(\lambda + 1/2) \Gamma(\nu - \lambda)} \int_{0}^{\theta} \sin^{2\lambda} \phi [\cos^2 \phi - \cos \phi \cos \theta]^{\nu - \lambda - 1} \frac{C_n^\lambda(\cos \phi)}{C_n(1)} \, d\phi 
\]

which is equivalent to Feldheim-Vilenkin formula (1.4); see Askey [1, pp. 23–24]. The evaluation of the limit of \( K(\cos \theta, \cos \phi; | q) \) as \( q \to 1 \) is a bit tricky, but similar calculations were done in Nassrallah and Rahman [16].
REFERENCES