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A PROJECTION FORMULA FOR THE ASKEY-WILSON POLYNOMIALS AND AN APPLICATION

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ABSTRACT. A projection formula for $p_n(x; a, b, c, d|q)$, the Askey-Wilson polynomials, is obtained by using a generalization of Askey and Wilson's q-beta integral. The result is used to find a q-analogue of the Feldheim-Vilenkin formula for ultraspherical polynomials. A q-analogue of the ultraspherical polynomials, other than the one due to Rogers, is also introduced.

1. Introduction. The main objective of this paper is to find a q-analogue of the projection formula

(1.1)

$$\frac{(1-x)^{\alpha+\mu}}{(1+x)^{n+\alpha+1}} \frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,\beta)}(1)} = \frac{2^\mu \Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha (y-x)^{\mu-1}}{(1+y)^{n+\alpha+\mu+1}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}\, dy,$$

$-1 < x < 1$, $\alpha > -1$, $\mu > 0$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{1.2}$$

A change of variable on both sides of (1.1) gives the accompanying formula

(1.3)

$$\frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} \frac{P_n^{(\alpha,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha)}(1)} = \frac{2^\mu \Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^x \frac{(1+y)^\beta (x-y)^{\mu-1}}{(1-y)^{n+\beta+\mu+1}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)}\, dy,$$

$-1 < x < 1$, $\beta > -1$, $\mu > 0$. Askey and Fitch [**3**] used (1.1) to give a simple proof of Feldheim [**10**] and Vilenkin's [**23**] formula

(1.4)

$$\begin{aligned} \frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)} = {} & \frac{2\Gamma(\nu+1/2)}{\Gamma(\lambda+1/2)\Gamma(\nu-\lambda)} \int_0^{\pi/2} \sin^{2\lambda}\phi \cos^{2\nu-2\lambda-1}\phi (1-\sin^2\theta\cos^2\phi)^{n/2} \\ & \cdot \frac{C_n^\lambda(\cos\theta(1-\sin^2\theta\cos^2\phi)^{-1/2})}{C_n^\lambda(1)}\, d\phi, \qquad \nu > \lambda > -\frac{1}{2}, \end{aligned}$$

where

$$\begin{aligned} C_n^\lambda(\cos\theta) &= \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(\cos\theta) \\ &= \sum_{k=0}^n \frac{(\lambda)_k (\lambda)_{n-k}}{k!(n-k)!} \cos(n-2k)\theta \end{aligned} \tag{1.5}$$

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is the ultraspherical polynomial; see Szegö [**22**]. Feldheim [**10**] used (1.4) and the positivity of Fejér's [**9**] sum $\sum_{k=0}^{n} P_k(x)$ for Legendre polynomials to show that

$$\sum_{k=0}^{n} \frac{C_k^\nu(\cos\theta)}{C_k^\nu(1)} > 0, \qquad 0 \le \theta < \pi,\ \nu \ge \frac{1}{2}. \tag{1.6}$$

Askey and Fitch [**3**] used (1.4) to deduce the integral representation

$$\frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = \int_{-1}^{1} \frac{P_n^{(\beta,\beta)}(y)}{P_n^{(\beta,\beta)}(1)}\, d\mu(y), \tag{1.7}$$

where $\alpha > \beta \ge -1/2$, $-1 \le x \le 1$, and $d\mu(y)$ is a positive measure which depends on x but not on n.

Askey [**2**] also gave an elementary proof of Koornwinder's [**12**, **13**] Laplace-type integral representation for Jacobi polynomials by using (1.1). Laine [**15**] found an extension of (1.1) to obtain a formula that is equivalent to Koornwinder's [**12**, **14**] addition formula for Jacobi polynomials.

To find a q-analogue of (1.1) we shall need the following extension of the beta-integral

$$\begin{aligned}&\int_0^\pi \left\{ (e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos\theta, a_1a_2a_3a_4a_5) \Big/ \prod_{j=1}^{5} h(\cos\theta, a_j) \right\} d\theta \\ &\quad = 2\pi \prod_{j=1}^{5} (a_1a_2a_3a_4a_5/a_j; q)_\infty / (q;q)_\infty \prod_{1\le j<k\le 5} (a_ja_k; q)_\infty,\end{aligned} \tag{1.8}$$

where

$$(A;q)_\infty = \prod_{j=0}^{\infty} (1 - Aq^j), \qquad 0 < q < 1, \tag{1.9}$$

$$h(\cos\theta, a) = \prod_{j=0}^{\infty} (1 - 2a\cos\theta q^j + a^2q^{2j}) = (ae^{i\theta}; q)_\infty (ae^{-i\theta}; q)_\infty, \tag{1.10}$$

and $\max(|a_j|) < 1$, $j = 1, \ldots, 5$. If we set $a_5 = 0$ then (1.8) reduces to the q-beta integral of Askey and Wilson [**6**, (2.1)]. Formula (1.8) was given in Rahman [**17**].

Askey and Wilson [**6**] introduced the polynomials

$$\begin{aligned}p_n(x) &= p_n(x; a, b, c, d|q) \\ &= {}_4\phi_3 \left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\theta}, & ae^{-i\theta} \\ & ab, & ac, & ad \end{matrix} ; q, q \right], \qquad x = \cos\theta,\end{aligned} \tag{1.11}$$

as a generalization of the Jacobi polynomials and showed that they are orthogonal on $[-1, 1]$ with respect to the weight function

$$\begin{aligned}W(x) &\equiv W(x; a, b, c, d|q) \\ &= \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)} (1 - x^2)^{-1/2}\end{aligned} \tag{1.12}$$

where a, b, c, d are either real or, if complex, occur in conjugate pairs such that their absolute values are less than 1. We shall assume throughout this paper that $0 < q < 1$. The symbol on the right side of (1.11) represents a basic hypergeometric series defined by

$$(1.13)\qquad {}_{r+1}\phi_r\left[\begin{matrix} a_1, & a_2, \dots, a_{r+1} \\ & b_1, \dots, b_r \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n,$$

where

$$(1.14)\qquad \begin{aligned} &(A_1, A_2, \dots, A_m; q)_n = \prod_{j=1}^{m} (A_j; q)_n, \\ &(A; q)_n = \begin{cases} 1, & n = 0, \\ (1-A)(1-Aq)\cdots(1-Aq^{n-1}), & n = 1, 2, \dots . \end{cases} \end{aligned}$$

If any one of the parameters $a_1, \dots, a_{r+1}$ is of the form q^{-k}, k a nonnegative integer, then the series in (1.13) terminates, otherwise we require that $|z| < 1$ to ensure convergence. If $z = q$ and $b_1 \cdots b_r = qa_1 \cdots a_{r+1}$, then the ${}_{r+1}\phi_r$ series is said to be balanced. If $qa_1 \neq a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r$ then the series is called a nearly-poised series of the first kind, and it is a nearly-poised series of the second kind if $qa_1 = a_2b_1 = \cdots \neq a_{r+1}b_r$. The ${}_{r+1}\phi_r$ series is called well-poised if $qa_1 = a_2b_1 = \cdots = a_{r+1}b_r$; if, in addition, $a_2 = -a_3 = qa_1^{1/2}$ then we call the series very-well-poised. Note that the ${}_4\phi_3$ series in (1.11) is balanced and terminating.

Since there are four parameters in $p_n(x; a, b, c, d|q)$ there are many ways to choose them so that a continuous q-Jacobi polynomial can be defined in the sense that $p_n(x)$ gives $P_n^{(\alpha,\beta)}(x)$ in the limit $q \to 1$. Askey and Wilson [**6**] took

$$(1.15)\qquad a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\beta+1)/4}, \quad d = -q^{(2\beta+3)/4}$$

to define

$$(1.16)\qquad P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{(2\alpha+1)/4}e^{i\theta}, & q^{(2\alpha+1)/4}e^{-i\theta} \\ & q^{\alpha+1}, & -q^{(\alpha+\beta+1)/2}, & -q^{(\alpha+\beta+2)/2} \end{matrix} ; q, q\right]$$

as a q-analogue of the Jacobi polynomials. The author [**18**] chose

$$(1.17)\qquad a = q^{1/2}, \quad b = q^{\alpha+1/2}, \quad c = -q^{\beta+1/2}, \quad d = -q^{1/2}$$

to introduce

$$(1.18)\qquad P_n^{(\alpha,\beta)}(x; q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q, -q; q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{1/2}e^{i\theta}, & q^{1/2}e^{-i\theta} \\ & q^{\alpha+1}, & -q^{\beta+1}, & -q \end{matrix} ; q, q\right]$$

as another q-analogue. It is easy to see that

$$(1.19)\qquad \lim_{q\to 1} P_n^{(\alpha,\beta)}(x|q) = \lim_{q\to 1} P_n^{(\alpha,\beta)}(x; q) = P_n^{(\alpha,\beta)}(x).$$

Also, as Askey and Wilson [**6**] pointed out, these analogoues are related by the formula

$$(1.20)\qquad P_n^{(\alpha,\beta)}(x|q^2) = \frac{(-q; q)_n}{(-q^{\alpha+\beta+1}; q)_n} q^{\alpha n} P_n^{(\alpha,\beta)}(x; q).$$

A number of important results for Jacobi polynomials have already been extended by using one or the other of these q-extensions (see, for example, Askey and Wilson **[6]**, Gasper and Rahman **[11]**, Rahman **[19]**), including a projection formula for $P_n^{(\alpha,\beta)}(x|q)$; see Nassrallah and Rahman **[16]**. However, we were unable to derive a q-analogue of (1.1) by choosing the parameters as in (1.15) or (1.17). So we were forced to consider other possibilities and we found that a projection formula analogous to (1.1) exists if we take

$$a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\alpha+1)/4}, \quad d = -q^{\beta+(3-2\alpha)/4}. \tag{1.21}$$

The basic formula that we shall prove in §2 is

$$\begin{aligned}&\int_{-1}^{1} v(y; a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(abc\mu e^{i\theta}, abc\mu e^{-i\theta}; q)_n}{(abc\mu^2 e^{i\phi}, abc\mu^2 e^{-i\phi}; q)_n} p_n(y; a, b, c, d|q)\, dy \\ &\qquad = g(a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}|q) \frac{(bc; q)_n}{(bc\mu^2; q)_n} p_n(x; a\mu, b\mu, c\mu, d\mu^{-1}|q)\end{aligned} \tag{1.22}$$

where $x = \cos\theta$, $y = \cos\phi$ and

$$\begin{aligned}&v(y; a_1, a_2, a_3, a_4, a_5|q) \\ &\qquad = \frac{h(y,1)h(y,-1)h(y,q^{1/2})h(y,-q^{1/2})h(y,a_1a_2a_3a_4a_5)}{h(y,a_1)h(y,a_2)h(y,a_3)h(y,a_4)h(y,a_5)}(1-y^2)^{-1/2},\end{aligned} \tag{1.23}$$

$$\begin{aligned}g(a_1, a_2, a_3, a_4, a_5|q) &= \int_{-1}^{1} v(y; a_1, a_2, a_3, a_4, a_5|q)\, dy \\ &= 2\pi \prod_{j=1}^{5} \left(\frac{a_1a_2a_3a_4a_5}{a_j}; q\right)_\infty \Big/ (q;q)_\infty \prod_{1\le j<k\le 5} (a_ja_k; q)_\infty,\end{aligned} \tag{1.24}$$

by (1.8). In (1.22) it is assumed that $-1 < a, b, c < 1$ and $0 < \mu < 1$. It is not hard to see that (1.22) is indeed a q-analogue of (1.1) if we choose the parameters as given in (1.21).

In §3 we shall derive the following q-analogue of the ultraspherical polynomials

$$\begin{aligned}&G_n(\cos\theta; \beta|q) \\ &= \frac{(-\beta^{1/2}q^{(n+1)/2}e^{-2i\theta}, -\beta^{1/2}q^{(1-n)/2}e^{2i\theta}; q)_\infty}{(-\beta^{1/2}q^{(n+1)/2-[n/2]}e^{-2i\theta}, -\beta^{1/2}q^{(1-n)/2+[n/2]}e^{2i\theta}; q)_\infty} e^{2i(n/2-[n/2])\theta} \\ &\quad \cdot \frac{(\beta;q)_n}{(q;q)_n} {}_3\phi_2\left[\begin{matrix} q^{-n}, & \beta, & -\beta^{-1/2}q^{(1-n)/2}e^{-2i\theta} \\ & \beta^{-1}q^{1-n}, & -\beta^{1/2}q^{(1-n)/2}e^{2i\theta}\end{matrix}; q, q\right]\end{aligned} \tag{1.25}$$

which is quite different and much more complicated than the q-ultraspherical polynomials of Rogers (see Askey and Ismail **[4]**). Formulas (3.5) and (3.7) will show, respectively, that $G_n(\cos\theta;\beta|q)$ is a polynomial when n is even, but not so when n is odd unless it is divided by $e^{i\theta} + \beta^{1/2}e^{-i\theta}$. However, it has the property that

$$\lim_{q\to 1} G_n(\cos\theta; q^\lambda|q) = C_n^\lambda(\cos\theta), \qquad \lambda > 0 \tag{1.26}$$

and that there is a q-analogue of the Feldheim-Vilenkin formula (1.4) for $G_n(\cos\theta;\beta|q)$, which we will derive in §4.

2. Proof of (1.22). Let j, n be nonnegative integers with $j \le n$ and let a, b, μ, ν be complex numbers with modulus less than 1. Also let $0 \le \theta \le \pi$. Then, by (1.8) and (1.24)

(2.1)
$$\begin{aligned}
&\int_{-1}^{1} v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab\nu\mu^2 e^{i\phi}, ab\nu\mu^2 e^{-i\phi}; q)_n}\, dy \\
&= \int_{-1}^{1} v(y; aq^j, bq^{n-j}, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \\
&= g(aq^j, bq^{n-j}, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \\
&= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) q^{j(j+1)-2nj} b^{-2j} \\
&\quad \cdot \frac{(ab, b\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n (a\nu, q^{1-n}/b\nu\mu^2, a\mu e^{i\theta}, a\mu e^{-i\theta}; q)_j}{(ab\mu^2, b\nu\mu^2, ab\mu\nu e^{i\theta}, ab\mu\nu e^{-i\theta}; q)_n (q^{1-n}/b\nu, a\nu\mu^2, q^{1-n}e^{-i\theta}/b\mu, q^{1-n}e^{i\theta}/b\mu; q)_j}.
\end{aligned}$$

Since, by Watson's formula [**7**, 8.5 (2)],

(2.2)
$$\begin{aligned}
&{}_4\phi_3 \left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\phi}, & ae^{-i\phi} \\ & ab, & ac, & ad \end{matrix} ; q, q \right] \\
&= \frac{(be^{i\theta}, be^{-i\theta}; q)_n}{(ab, b/a; q)_n} {}_8W_7(aq^{-n}/b; ae^{i\theta}, ae^{-i\theta}, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q, cdq^n) \\
&= \sum_{j=0}^{n} \frac{(aq^{-n}/b, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q)_j (1 - aq^{2j-n}/b)}{(q, ac, ad, aq/b; q)_j (1 - aq^{-n}/b)} (b^2cd)^j q^{3nj - j(j+1)} \\
&\quad \cdot \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab, b/a; q)_n},
\end{aligned}$$

where

$$\begin{aligned}
&{}_{r+3}W_{r+2}(a; q_1, \ldots, a_r; q, z) \\
&\equiv {}_{r+3}\phi_{r+2} \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & a_1, \ldots, a_r \\ & \sqrt{a}, & -\sqrt{a}, & qa/a_1, \ldots, aq/a_r \end{matrix} ; q, z \right],
\end{aligned} \tag{2.3}$$

we find that

$$\begin{aligned}
&\int_{-1}^{1} v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{p_n(y; a, b, c, d|q)}{(ab\nu\mu^2 e^{i\phi}, ab\nu\mu^2 e^{-i\phi}; q)_n}\, dy \\
&= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(b\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n}{(b/a, ab\mu^2, b\nu\mu^2, ab\mu\nu e^{i\theta}, ab\mu\nu e^{-i\theta}; q)_n} \\
&\quad \cdot {}_{10}W_9(aq^{-n}/b; a\mu e^{i\theta}, a\mu e^{-i\theta}, a\nu, q^{1-n}/bc, q^{1-n}/bd, q^{1-n}/b\nu\mu^2, q^{-n}; q, cdq^n).
\end{aligned} \tag{2.4}$$

If we choose $\nu = c$ then the ${}_{10}W_9$ series on the right side becomes an ${}_8\phi_7$ series which, in turn, is expressible as $p_n(x)$ with a, b, c, d replaced by $a\mu, b\mu, c\mu, d\mu^{-1}$, respectively. This completes the proof of (1.22). In order that (1.22) be seen as a q-analogue of (1.1) we need to impose the additional restrictions mentioned earlier, i.e., $-1 < a, b, c < 1$ and $0 < \mu < 1$.

Transforming the ${}_4\phi_3$ series in (1.11) by Sears' transformation formula [**20**, (8.3)] and interchanging the parameters we obtain from (1.22) another formula

$$\begin{aligned}\int_{-1}^{1} v(y;b,c,d,\mu e^{i\theta},\mu e^{-i\theta}|q)\frac{(bcd\mu e^{i\theta},bcd\mu e^{-i\theta};q)_n}{(bcd\mu^2 e^{i\phi},bcd\mu^2 e^{-i\phi};q)_n}p_n(y;a,b,c,d|q)\,dy \\ = g(b,c,d,\mu e^{i\theta},\mu e^{-i\theta}|q)\frac{(bc,bd,cd;q)_n}{(bc\mu^2,bd\mu^2,cd\mu^2;q)_n}p_n(x;a\mu^{-1},b\mu,c\mu,d\mu|q)\end{aligned} \tag{2.5}$$

which can be seen as a q-analogue of (1.3) if we specialize the parameters as

$$a = q^{\alpha+(3-2\beta)/4},\quad b = q^{(2\beta+1)/4},\quad c = -q^{(2\beta+1)/4},\quad d = -q^{(2\beta+3)/4}. \tag{2.6}$$

3. A q-analogue of the ultraspherical polynomials. By (1.11) and [**7**, 8.5 (2)],

$$\begin{aligned} p_n(x;a,aq^{1/2},-a,-q^{1/2}/a|q) &= {}_4\phi_3\left[\begin{matrix} q^{-n}, & a^2q^n, & ae^{i\theta}, & ae^{-i\theta} \\ & a^2q^{1/2}, & -a^2, & -q^{1/2}\end{matrix};q,q\right] \\ &= \frac{(-aq^{1/2-n}e^{i\theta},-aq^{1/2-n}e^{-i\theta};q)_n}{(-a^2q^{1/2-n},-q^{1/2-n};q)_n} \\ &\quad\cdot {}_8W_7(-a^2q^{-n-1/2};ae^{i\theta},ae^{-i\theta},q^{1/2-n},-q^{-n},q^{-n};q,-a^2q^{n+1/2}).\end{aligned} \tag{3.1}$$

Setting $d = -(aq)^{1/2}$ in Bailey's formula [**21**, (3.4.1.6)] we find that

$$\begin{aligned} {}_4\phi_3\left[\begin{matrix} a, & b, & c, & q^{-n} \\ aq/b, & aq/c, & b^2c^2q^{-n-1}/a \end{matrix};q,q\right] &= \frac{(a^2q^2/b^2c^2,-a^{1/2}q^{3/2}/bc;q)_n}{(aq^2/b^2c^2,-a^{3/2}q^{3/2}/bc;q)_n} \\ \cdot\, {}_{10}W_9(-a^{3/2}q^{1/2}/bc;a^{1/2},-a^{1/2},(aq)^{1/2} &- (aq)^{1/2}/b, \\ &- (aq)^{1/2}/c,a^2q^{n+2}/b^2c^2,q^{-n};q,q)\end{aligned} \tag{3.2}$$

from which, it follows by taking the limit $n \to \infty$ and replacing a,b,c by q^{-2n}, $-q^{1/2-n}e^{i\theta}/a$ and $-q^{1/2-n}e^{-i\theta}/a$, respectively, that

$$\begin{aligned} &{}_8W_7(-a^2q^{-n-1/2};ae^{i\theta},ae^{-i\theta},q^{1/2-n},-q^{-n},q^{-n};q,-a^2q^{n+1/2}) \\ &\quad= \frac{(a^4q^{2n},-a^2q^{1/2-n};q)_\infty}{(a^4,-a^2q^{n+1/2};q)_\infty} \\ &\quad\times {}_3\phi_2\left[\begin{matrix} q^{-2n}, & -q^{1/2-n}e^{i\theta}/a, & -q^{1/2-n}e^{-i\theta}/a \\ & -aq^{1/2-n}e^{-i\theta}, & -aq^{1/2-n}e^{i\theta}\end{matrix};q,a^4q^{2n}\right].\end{aligned} \tag{3.3}$$

By Sears' transformation formula [**20**, II(a), p. 173],

$$\begin{aligned} &{}_3\phi_2\left[\begin{matrix} q^{-2n}, & -q^{1/2-n}e^{i\theta}/a, & -q^{1/2-n}e^{-i\theta}/a \\ & -aq^{1/2-n}e^{-i\theta}, & -aq^{1/2-n}e^{i\theta}\end{matrix};q,a^4q^{2n}\right] \\ &\quad= \frac{(a^2;q)_{2n}}{(-aq^{1/2-n}e^{-i\theta};q)_{2n}}{}_3\phi_2\left[\begin{matrix} q^{-2n}, & a^2, & -q^{1/2-n}e^{-i\theta}/a \\ & q^{1-2n}/a^2, & -aq^{1/2-n}e^{i\theta}\end{matrix};q,q\right]\end{aligned} \tag{3.4}$$

and hence by combining (3.1), (3.3) and (3.4) we find that

(3.5)
$$p_n(\cos 2\theta; a, aq^{1/2}, -a, -q^{1/2}/a|q) = \frac{(q;q)_{2n}}{(a^4;q)_{2n}} \frac{(-a^2q^{1/2};q)_n}{(-q^{1/2-n};q)_n} G_{2n}(\cos\theta; a^2|q),$$

where $G_n(x;\beta|q)$ is as defined in (1.25). This is a q-analogue of the quadratic transformation formula [**8**, 10.9 (21)]:

$$P_n^{(\lambda-1/2,-1/2)}(\cos 2\theta) = \frac{(1/2)_n}{(\lambda)_n} C_{2n}^{\lambda}(\cos\theta). \tag{3.6}$$

Similarly one can show that

$$\begin{aligned} &p_n(\cos 2\theta; a, aq^{1/2}, -aq^{1/2}, -q/a|q) \\ &\quad = \frac{(q;q)_{2n+1}}{(a^4;q)_{2n+1}} \frac{(-qa^2;q)_n}{(-q^{-n};q)_n} \frac{1+a^2}{e^{i\theta}+ae^{-i\theta}} G_{2n+1}(\cos\theta; a^2|q) \end{aligned} \tag{3.7}$$

which is a q-analogue of [**8**, 10.9 (22)]:

$$\cos\theta P_n^{(\lambda-1/2,1/2)}(\cos 2\theta) = \frac{(1/2)_{n+1}}{(\lambda)_{n+1}} C_{2n+1}^{\lambda}(\cos\theta). \tag{3.8}$$

It is clear from (3.5) and (3.7) that $G_n(x;q^\lambda|q)$ behaves like the ultraspherical polynomials $C_n^\lambda(x)$ in the limit $q \to 1$, but otherwise quite different from and not nearly as nice as Rogers' continuous q-ultraspherical polynomials

$$C_n(\cos\theta;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \cos(n-2k)\theta. \tag{3.9}$$

The orthogonality and other properties of $C_n(x;\beta|q)$ are by now well known, mainly through the works of Askey and Ismail [**4**, **5**] and most of these results have proved very useful. We do not expect the functions $G_n(x;\beta|q)$ to be half as useful. But the fact that they occur in a q-analogue of an important formula, namely (1.4), for ultraspherical polynomials, is probably reason enough to look at them a bit more closely, study their properties and look for other applications. This we plan to do in a later report.

4. A q-analogue of the Feldheim-Vilenkin formula.

From (1.22) we find that

$$\begin{aligned} &\int_0^{\pi/2} \rho(\cos 2\phi; a, b, c, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(abc\mu e^{2i\theta}, abc\mu e^{-2i\theta};q)_n}{(abc\mu^2 e^{2i\phi}, abc\mu^2 e^{-2i\phi};q)_n} \\ &\qquad\qquad \cdot p_n(\cos 2\phi; a,b,c,d|q)\, d\phi \\ &\quad = \frac{1}{2} g(a,b,c,\mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(bc;q)_n}{(bc\mu^2;q)_n} p_n(\cos 2\theta; a\mu, b\mu, c\mu, d\mu^{-1}|q), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &\rho(\cos 2\phi; a_1,a_2,a_3,a_4,a_5|q) \\ &\quad = \frac{h(\cos 2\phi,1)h(\cos 2\phi,-1)h(\cos 2\phi,q^{1/2})h(\cos 2\phi,-q^{1/2})h(\cos 2\phi,a_1a_2a_3a_4a_5)}{h(\cos 2\phi,a_1)h(\cos 2\phi,a_2)h(\cos 2\phi,a_3)h(\cos 2\phi,a_4)h(\cos 2\phi,a_5)}. \end{aligned} \tag{4.2}$$

Using (3.5) we then have

(4.3)

$$\int_0^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -a, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(-q^{1/2}a^3\mu e^{2i\theta}, -q^{1/2}q^3\mu e^{-2i\theta}; q)_n}{(-q^{1/2}a^3\mu^2 e^{2i\phi}, -q^{1/2}a^3\mu^2 e^{-2i\phi}; q)_n} \cdot G_{2n}(\cos\phi; a^2|q)\, d\phi$$

$$= \frac{1}{2} g(a, aq^{1/2}, -a, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(a^4; q)_{2n}}{(a^4\mu^4; q)_{2n}} G_{2n}(\cos\theta; a^2\mu^2|q).$$

Similarly, the use of (3.7) gives

(4.4)

$$\int_0^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -aq^{1/2}, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(-qa^3\mu e^{2i\theta}, -qa^3\mu e^{-2i\theta}; q)_n}{(-qa^3\mu^2 e^{2i\phi}, -qa^3\mu^2 e^{-2i\phi}; q)_n} \cdot \frac{e^{i\theta} + \mu a e^{-i\theta}}{e^{i\phi} + ae^{-i\phi}} G_{2n+1}(\cos\phi; a^2|q)\, d\phi$$

$$= \frac{1}{2} g(a, aq^{1/2}, -aq^{1/2}, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{1 + a^2\mu^2}{1 + a^2} \frac{(a^4; q)_{2n+1}}{(a^4\mu^4; q)_{2n+1}} \cdot G_{2n+1}(\cos\theta; a^2\mu^2|q).$$

Combining (4.3) and (4.4) we get the formula

$$\int_0^{\pi/2} K(\cos\theta, \cos\phi; a^2|q) \frac{h(\cos 2\phi, -a^3\mu^2 q^{(n+1)/2})}{h(\cos 2\theta, -a^3\mu q^{(n+1)/2})} \cdot \frac{(-aq^{(n+1)/2-[n/2]} e^{-2i\phi}, -aq^{(1-n)/2+[n/2]} e^{2i\phi}; q)_\infty}{(-a\mu q^{(n+1)/2-[n/2]} e^{-2i\theta}, -a\mu q^{(1-n)/2+[n/2]} e^{2i\theta}; q)_\infty} e^{2i(\theta-\phi)(n/2-[n/2])} \cdot G_n(\cos\phi; a^2|q)\, d\phi \tag{4.5}$$

$$= \frac{(a^4; q)_n}{(a^4\mu^4; q)_n} G_n(\cos\theta; a^2\mu^2|q),$$

where

$$K(\cos\theta, \cos\phi; a^2|q) = \frac{(q, a^4, \mu^2, a^2\mu^2; q)_\infty}{\pi(a^4\mu^4, a^2; q)_\infty} \cdot \frac{h(\cos 4\phi, 1) h(\cos 4\theta, a^2\mu^2)}{h(\cos 4\phi, a^2) h(\cos 2\phi, \mu e^{2i\theta}) h(\cos 2\phi, \mu e^{-2i\theta})}, \qquad |a| < 1,\ 0 < \mu < 1. \tag{4.6}$$

Replacing a, μ by $q^{\lambda/2}, q^{(\nu-\lambda)/2}$, $\nu > \lambda > -1/2$, it can be shown by using the q-gamma function

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad \Gamma(x) = \lim_{q\to 1} \Gamma_q(x), \tag{4.7}$$

that (4.5) reduces to

$$\frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)} \frac{\sin^{2\nu-1}\theta}{\cos^{n+2\lambda+1}\theta} = \frac{2\Gamma(\nu+1/2)}{\Gamma(\lambda+1/2)\Gamma(\nu-\lambda)} \int_0^\theta \sin^{2\lambda}\phi \frac{[\cos^2\phi - \cos^2\theta]^{\nu-\lambda-1}}{\cos^{n+2\nu}\phi} \frac{C_n^\lambda(\cos\phi)}{C_n^\lambda(1)}\, d\phi \tag{4.8}$$

which is equivalent to Feldheim-Vilenkin formula (1.4); see Askey [**1**, pp. 23–24]. The evaluation of the limit of $K(\cos\theta, \cos\phi; |q)$ as $q \to 1$ is a bit tricky, but similar calculations were done in Nassrallah and Rahman [**16**].

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