

A PROJECTION FORMULA FOR THE ASKEY-WILSON POLYNOMIALS AND AN APPLICATION

MIZAN RAHMAN

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ABSTRACT. A projection formula for $p_n(x; a, b, c, d|q)$, the Askey-Wilson polynomials, is obtained by using a generalization of Askey and Wilson's q -beta integral. The result is used to find a q -analogue of the Feldheim-Vilenkin formula for ultraspherical polynomials. A q -analogue of the ultraspherical polynomials, other than the one due to Rogers, is also introduced.

1. Introduction. The main objective of this paper is to find a q -analogue of the projection formula

$$(1.1) \quad \frac{(1-x)^{\alpha+\mu} P_n^{(\alpha+\mu, \beta)}(x)}{(1+x)^{n+\alpha+1} P_n^{(\alpha+\mu, \beta)}(1)} = \frac{2^\mu \Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha (y-x)^{\mu-1} P_n^{(\alpha, \beta)}(y)}{(1+y)^{n+\alpha+\mu+1} P_n^{(\alpha, \beta)}(1)} dy,$$

$-1 < x < 1$, $\alpha > -1$, $\mu > 0$, where $P_n^{(\alpha, \beta)}(x)$ is the Jacobi polynomial defined by

$$(1.2) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2} \right).$$

A change of variable on both sides of (1.1) gives the accompanying formula

$$(1.3) \quad \frac{(1+x)^{\beta+\mu} P_n^{(\alpha, \beta+\mu)}(x)}{(1-x)^{n+\beta+1} P_n^{(\alpha, \beta+\mu)}(1)} = \frac{2^\mu \Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^x \frac{(1+y)^\beta (x-y)^{\mu-1} P_n^{(\alpha, \beta)}(y)}{(1-y)^{n+\beta+\mu+1} P_n^{(\alpha, \beta)}(1)} dy,$$

$-1 < x < 1$, $\beta > -1$, $\mu > 0$. Askey and Fitch [3] used (1.1) to give a simple proof of Feldheim [10] and Vilenkin's [23] formula

$$(1.4) \quad \frac{C_n^\nu(\cos \theta)}{C_n^\nu(1)} = \frac{2\Gamma(\nu+1/2)}{\Gamma(\lambda+1/2)\Gamma(\nu-\lambda)} \int_0^{\pi/2} \frac{\sin^{2\lambda} \phi \cos^{2\nu-2\lambda-1} \phi (1-\sin^2 \theta \cos^2 \phi)^{n/2}}{C_n^\lambda(\cos \theta (1-\sin^2 \theta \cos^2 \phi)^{-1/2})} d\phi, \quad \nu > \lambda > -\frac{1}{2},$$

where

$$(1.5) \quad \begin{aligned} C_n^\lambda(\cos \theta) &= \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2, \lambda-1/2)}(\cos \theta) \\ &= \sum_{k=0}^n \frac{(\lambda)_k (\lambda)_{n-k}}{k!(n-k)!} \cos(n-2k)\theta \end{aligned}$$

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is the ultraspherical polynomial; see Szegő [22]. Feldheim [10] used (1.4) and the positivity of Fejér's [9] sum $\sum_{k=0}^n P_k(x)$ for Legendre polynomials to show that

$$(1.6) \quad \sum_{k=0}^n \frac{C_k^\nu(\cos \theta)}{C_k^\nu(1)} > 0, \quad 0 \leq \theta < \pi, \nu \geq \frac{1}{2}.$$

Askey and Fitch [3] used (1.4) to deduce the integral representation

$$(1.7) \quad \frac{P_n^{(\alpha, \alpha)}(x)}{P_n^{(\alpha, \alpha)}(1)} = \int_{-1}^1 \frac{P_n^{(\beta, \beta)}(y)}{P_n^{(\beta, \beta)}(1)} d\mu(y),$$

where $\alpha > \beta \geq -1/2$, $-1 \leq x \leq 1$, and $d\mu(y)$ is a positive measure which depends on x but not on n .

Askey [2] also gave an elementary proof of Koornwinder's [12, 13] Laplace-type integral representation for Jacobi polynomials by using (1.1). Laine [15] found an extension of (1.1) to obtain a formula that is equivalent to Koornwinder's [12, 14] addition formula for Jacobi polynomials.

To find a q -analogue of (1.1) we shall need the following extension of the beta-integral

$$(1.8) \quad \int_0^\pi \left\{ (e^{2i\theta}, e^{-2i\theta}; q)_\infty h(\cos \theta, a_1 a_2 a_3 a_4 a_5) / \prod_{j=1}^5 h(\cos \theta, a_j) \right\} d\theta \\ = 2\pi \prod_{j=1}^5 (a_1 a_2 a_3 a_4 a_5 / a_j; q)_\infty / (q; q)_\infty \prod_{1 \leq j < k \leq 5} (a_j a_k; q)_\infty,$$

where

$$(1.9) \quad (A; q)_\infty = \prod_{j=0}^\infty (1 - Aq^j), \quad 0 < q < 1,$$

$$(1.10) \quad h(\cos \theta, a) = \prod_{j=0}^\infty (1 - 2a \cos \theta q^j + a^2 q^{2j}) = (ae^{i\theta}; q)_\infty (ae^{-i\theta}; q)_\infty,$$

and $\max(|a_j|) < 1$, $j = 1, \dots, 5$. If we set $a_5 = 0$ then (1.8) reduces to the q -beta integral of Askey and Wilson [6, (2.1)]. Formula (1.8) was given in Rahman [17].

Askey and Wilson [6] introduced the polynomials

$$(1.11) \quad p_n(x) = p_n(x; a, b, c, d|q) \\ = {}_4\phi_3 \left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\theta}, & ae^{-i\theta} \\ & ab, & ac, & ad \end{matrix} ; q, q \right], \quad x = \cos \theta,$$

as a generalization of the Jacobi polynomials and showed that they are orthogonal on $[-1, 1]$ with respect to the weight function

$$(1.12) \quad W(x) \equiv W(x; a, b, c, d|q) \\ = \frac{h(x, 1)h(x, -1)h(x, q^{1/2})h(x, -q^{1/2})}{h(x, a)h(x, b)h(x, c)h(x, d)} (1 - x^2)^{-1/2}$$

where a, b, c, d are either real or, if complex, occur in conjugate pairs such that their absolute values are less than 1. We shall assume throughout this paper that $0 < q < 1$. The symbol on the right side of (1.11) represents a basic hypergeometric series defined by

$$(1.13) \quad {}_{r+1}\phi_r \left[\begin{matrix} a_1, & a_2, \dots, a_{r+1} \\ & b_1, \dots, b_r \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n,$$

where

$$(1.14) \quad (A_1, A_2, \dots, A_m; q)_n = \prod_{j=1}^m (A_j; q)_n,$$

$$(A; q)_n = \begin{cases} 1, & n = 0, \\ (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}), & n = 1, 2, \dots \end{cases}$$

If any one of the parameters a_1, \dots, a_{r+1} is of the form q^{-k} , k a nonnegative integer, then the series in (1.13) terminates, otherwise we require that $|z| < 1$ to ensure convergence. If $z = q$ and $b_1 \cdots b_r = qa_1 \cdots a_{r+1}$, then the ${}_{r+1}\phi_r$ series is said to be balanced. If $qa_1 \neq a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r$ then the series is called a nearly-poised series of the first kind, and it is a nearly-poised series of the second kind if $qa_1 = a_2b_1 = \cdots \neq a_{r+1}b_r$. The ${}_{r+1}\phi_r$ series is called well-poised if $qa_1 = a_2b_1 = \cdots = a_{r+1}b_r$; if, in addition, $a_2 = -a_3 = qa_1^{1/2}$ then we call the series very-well-poised. Note that the ${}_4\phi_3$ series in (1.11) is balanced and terminating.

Since there are four parameters in $p_n(x; a, b, c, d|q)$ there are many ways to choose them so that a continuous q -Jacobi polynomial can be defined in the sense that $p_n(x)$ gives $P_n^{(\alpha, \beta)}(x)$ in the limit $q \rightarrow 1$. Askey and Wilson [6] took

$$(1.15) \quad a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\beta+1)/4}, \quad d = -q^{(2\beta+3)/4}$$

to define

$$(1.16) \quad P_n^{(\alpha, \beta)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{(2\alpha+1)/4}e^{i\theta}, & q^{(2\alpha+1)/4}e^{-i\theta} \\ & q^{\alpha+1}, & -q^{(\alpha+\beta+1)/2}, & -q^{(\alpha+\beta+2)/2} \end{matrix} ; q, q \right]$$

as a q -analogue of the Jacobi polynomials. The author [18] chose

$$(1.17) \quad a = q^{1/2}, \quad b = q^{\alpha+1/2}, \quad c = -q^{\beta+1/2}, \quad d = -q^{1/2}$$

to introduce

$$(1.18) \quad P_n^{(\alpha, \beta)}(x; q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q, -q; q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, & q^{n+\alpha+\beta+1}, & q^{1/2}e^{i\theta}, & q^{1/2}e^{-i\theta} \\ & q^{\alpha+1}, & -q^{\beta+1}, & -q \end{matrix} ; q, q \right]$$

as another q -analogue. It is easy to see that

$$(1.19) \quad \lim_{q \rightarrow 1} P_n^{(\alpha, \beta)}(x|q) = \lim_{q \rightarrow 1} P_n^{(\alpha, \beta)}(x; q) = P_n^{(\alpha, \beta)}(x).$$

Also, as Askey and Wilson [6] pointed out, these analogues are related by the formula

$$(1.20) \quad P_n^{(\alpha, \beta)}(x|q^2) = \frac{(-q; q)_n}{(-q^{\alpha+\beta+1}; q)_n} q^{\alpha n} P_n^{(\alpha, \beta)}(x; q).$$

A number of important results for Jacobi polynomials have already been extended by using one or the other of these q -extensions (see, for example, Askey and Wilson [6], Gasper and Rahman [11], Rahman [19]), including a projection formula for $P_n^{(\alpha,\beta)}(x|q)$; see Nassrallah and Rahman [16]. However, we were unable to derive a q -analogue of (1.1) by choosing the parameters as in (1.15) or (1.17). So we were forced to consider other possibilities and we found that a projection formula analogous to (1.1) exists if we take

$$(1.21) \quad a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\alpha+1)/4}, \quad d = -q^{\beta+(3-2\alpha)/4}.$$

The basic formula that we shall prove in §2 is

$$(1.22) \quad \int_{-1}^1 v(y; a, b, c, \mu e^{i\theta}, \mu e^{-i\theta} | q) \frac{(abc\mu e^{i\theta}, abc\mu e^{-i\theta}; q)_n}{(abc\mu^2 e^{i\phi}, abc\mu^2 e^{-i\phi}; q)_n} p_n(y; a, b, c, d | q) dy \\ = g(a, b, c, \mu e^{i\theta}, \mu e^{-i\theta} | q) \frac{(bc; q)_n}{(bc\mu^2; q)_n} p_n(x; a\mu, b\mu, c\mu, d\mu^{-1} | q)$$

where $x = \cos \theta, y = \cos \phi$ and

$$(1.23) \quad v(y; a_1, a_2, a_3, a_4, a_5 | q) \\ = \frac{h(y, 1)h(y, -1)h(y, q^{1/2})h(y, -q^{1/2})h(y, a_1 a_2 a_3 a_4 a_5)}{h(y, a_1)h(y, a_2)h(y, a_3)h(y, a_4)h(y, a_5)} (1 - y^2)^{-1/2},$$

$$(1.24) \quad g(a_1, a_2, a_3, a_4, a_5 | q) = \int_{-1}^1 v(y; a_1, a_2, a_3, a_4, a_5 | q) dy \\ = 2\pi \prod_{j=1}^5 \left(\frac{a_1 a_2 a_3 a_4 a_5}{a_j}; q \right)_{\infty} / (q; q)_{\infty} \prod_{1 \leq j < k \leq 5} (a_j a_k; q)_{\infty},$$

by (1.8). In (1.22) it is assumed that $-1 < a, b, c < 1$ and $0 < \mu < 1$. It is not hard to see that (1.22) is indeed a q -analogue of (1.1) if we choose the parameters as given in (1.21).

In §3 we shall derive the following q -analogue of the ultraspherical polynomials

$$(1.25) \quad G_n(\cos \theta; \beta | q) \\ = \frac{(-\beta^{1/2} q^{(n+1)/2} e^{-2i\theta}, -\beta^{1/2} q^{(1-n)/2} e^{2i\theta}; q)_{\infty}}{(-\beta^{1/2} q^{(n+1)/2 - [n/2]} e^{-2i\theta}, -\beta^{1/2} q^{(1-n)/2 + [n/2]} e^{2i\theta}; q)_{\infty}} e^{2i(n/2 - [n/2])\theta} \\ \cdot \frac{(\beta; q)_n}{(q; q)_n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, & \beta, & -\beta^{-1/2} q^{(1-n)/2} e^{-2i\theta} \\ & \beta^{-1} q^{1-n}, & -\beta^{1/2} q^{(1-n)/2} e^{2i\theta} \end{matrix}; q, q \right]$$

which is quite different and much more complicated than the q -ultraspherical polynomials of Rogers (see Askey and Ismail [4]). Formulas (3.5) and (3.7) will show, respectively, that $G_n(\cos \theta; \beta | q)$ is a polynomial when n is even, but not so when n is odd unless it is divided by $e^{i\theta} + \beta^{1/2} e^{-i\theta}$. However, it has the property that

$$(1.26) \quad \lim_{q \rightarrow 1} G_n(\cos \theta; q^{\lambda} | q) = C_n^{\lambda}(\cos \theta), \quad \lambda > 0$$

and that there is a q -analogue of the Feldheim-Vilenkin formula (1.4) for $G_n(\cos \theta; \beta | q)$, which we will derive in §4.

2. Proof of (1.22). Let j, n be nonnegative integers with $j \leq n$ and let a, b, μ, ν be complex numbers with modulus less than 1. Also let $0 \leq \theta \leq \pi$. Then, by (1.8) and (1.24)

$$\begin{aligned}
 (2.1) \quad & \int_{-1}^1 v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab\nu\mu^2 e^{i\phi}, ab\nu\mu^2 e^{-i\phi}; q)_n} dy \\
 &= \int_{-1}^1 v(y; aq^j, bq^{n-j}, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \\
 &= g(aq^j, bq^{n-j}, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \\
 &= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) q^{j(j+1)-2nj} b^{-2j} \\
 &\quad \cdot \frac{(ab, b\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n (a\nu, q^{1-n}/b\nu\mu^2, a\mu e^{i\theta}, a\mu e^{-i\theta}; q)_j}{(ab\mu^2, b\nu\mu^2, ab\mu\nu e^{i\theta}, ab\mu\nu e^{-i\theta}; q)_n (q^{1-n}/b\nu, a\nu\mu^2, q^{1-n}e^{-i\theta}/b\mu, q^{1-n}e^{i\theta}/b\mu; q)_j}.
 \end{aligned}$$

Since, by Watson’s formula [7, 8.5 (2)],

$$\begin{aligned}
 (2.2) \quad & 4\phi_3 \left[\begin{matrix} q^{-n}, & abcdq^{n-1}, & ae^{i\phi}, & ae^{-i\phi} \\ & ab, & ac, & ad \end{matrix} ; q, q \right] \\
 &= \frac{(be^{i\theta}, be^{-i\theta}; q)_n}{(ab, b/a; q)_n} {}_8W_7(aq^{-n}/b; ae^{i\theta}, ae^{-i\theta}, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q, cdq^n) \\
 &= \sum_{j=0}^n \frac{(aq^{-n}/b, q^{1-n}/bc, q^{1-n}/bd, q^{-n}; q)_j (1 - aq^{2j-n}/b)}{(q, ac, ad, aq/b; q)_j (1 - aq^{-n}/b)} (b^2 cd)^j q^{3nj-j(j+1)} \\
 &\quad \cdot \frac{(ae^{i\phi}, ae^{-i\phi}; q)_j (be^{i\phi}, be^{-i\phi}; q)_{n-j}}{(ab, b/a; q)_n},
 \end{aligned}$$

where

$$\begin{aligned}
 (2.3) \quad & {}_{r+3}W_{r+2}(a; q_1, \dots, a_r; q, z) \\
 &\equiv {}_{r+3}\phi_{r+2} \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & a_1, \dots, a_r \\ & \sqrt{a}, & -\sqrt{a}, & qa/a_1, \dots, aq/a_r \end{matrix} ; q, z \right],
 \end{aligned}$$

we find that

$$\begin{aligned}
 (2.4) \quad & \int_{-1}^1 v(y; a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{p_n(y; a, b, c, d|q)}{(ab\nu\mu^2 e^{i\phi}, ab\nu\mu^2 e^{-i\phi}; q)_n} dy \\
 &= g(a, b, \mu e^{i\theta}, \mu e^{-i\theta}, \nu|q) \frac{(b\nu, b\mu e^{i\theta}, b\mu e^{-i\theta}; q)_n}{(b/a, ab\mu^2, b\nu\mu^2, ab\mu\nu e^{i\theta}, ab\mu\nu e^{-i\theta}; q)_n} \\
 &\quad \cdot {}_{10}W_9(aq^{-n}/b; a\mu e^{i\theta}, a\mu e^{-i\theta}, a\nu, q^{1-n}/bc, q^{1-n}/bd, q^{1-n}/b\nu\mu^2, q^{-n}; q, cdq^n).
 \end{aligned}$$

If we choose $\nu = c$ then the ${}_{10}W_9$ series on the right side becomes an ${}_8\phi_7$ series which, in turn, is expressible as $p_n(x)$ with a, b, c, d replaced by $a\mu, b\mu, c\mu, d\mu^{-1}$, respectively. This completes the proof of (1.22). In order that (1.22) be seen as a q -analogue of (1.1) we need to impose the additional restrictions mentioned earlier, i.e., $-1 < a, b, c < 1$ and $0 < \mu < 1$.

Transforming the ${}_4\phi_3$ series in (1.11) by Sears' transformation formula [20, (8.3)] and interchanging the parameters we obtain from (1.22) another formula

$$(2.5) \quad \int_{-1}^1 v(y; b, c, d, \mu e^{i\theta}, \mu e^{-i\theta} | q) \frac{(bcd\mu e^{i\theta}, bcd\mu e^{-i\theta}; q)_n}{(bcd\mu^2 e^{i\phi}, bcd\mu^2 e^{-i\phi}; q)_n} p_n(y; a, b, c, d | q) dy \\ = g(b, c, d, \mu e^{i\theta}, \mu e^{-i\theta} | q) \frac{(bc, bd, cd; q)_n}{(bc\mu^2, bd\mu^2, cd\mu^2; q)_n} p_n(x; a\mu^{-1}, b\mu, c\mu, d\mu | q)$$

which can be seen as a q -analogue of (1.3) if we specialize the parameters as

$$(2.6) \quad a = q^{\alpha+(3-2\beta)/4}, \quad b = q^{(2\beta+1)/4}, \quad c = -q^{(2\beta+1)/4}, \quad d = -q^{(2\beta+3)/4}.$$

3. A q -analogue of the ultraspherical polynomials. By (1.11) and [7, 8.5 (2)],

$$(3.1) \quad p_n(x; a, aq^{1/2}, -a, -q^{1/2}/a | q) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, & a^2q^n, & ae^{i\theta}, & ae^{-i\theta} \\ & a^2q^{1/2}, & -a^2, & -q^{1/2} \end{matrix} ; q, q \right] \\ = \frac{(-aq^{1/2-n}e^{i\theta}, -aq^{1/2-n}e^{-i\theta}; q)_n}{(-a^2q^{1/2-n}, -q^{1/2-n}; q)_n} \\ \cdot {}_8W_7(-a^2q^{-n-1/2}; ae^{i\theta}, ae^{-i\theta}, q^{1/2-n}, -q^{-n}, q^{-n}; q, -a^2q^{n+1/2}).$$

Setting $d = -(aq)^{1/2}$ in Bailey's formula [21, (3.4.1.6)] we find that

$$(3.2) \quad {}_4\phi_3 \left[\begin{matrix} a, & b, & c, & q^{-n} \\ & aq/b, & aq/c, & b^2c^2q^{-n-1}/a \end{matrix} ; q, q \right] = \frac{(a^2q^2/b^2c^2, -a^{1/2}q^{3/2}/bc; q)_n}{(aq^2/b^2c^2, -a^{3/2}q^{3/2}/bc; q)_n} \\ \cdot {}_{10}W_9(-a^{3/2}q^{1/2}/bc; a^{1/2}, -a^{1/2}, (aq)^{1/2} - (aq)^{1/2}/b, \\ - (aq)^{1/2}/c, a^2q^{n+2}/b^2c^2, q^{-n}; q, q)$$

from which, it follows by taking the limit $n \rightarrow \infty$ and replacing a, b, c by $q^{-2n}, -q^{1/2-n}e^{i\theta}/a$ and $-q^{1/2-n}e^{-i\theta}/a$, respectively, that

$$(3.3) \quad {}_8W_7(-a^2q^{-n-1/2}; ae^{i\theta}, ae^{-i\theta}, q^{1/2-n}, -q^{-n}, q^{-n}; q, -a^2q^{n+1/2}) \\ = \frac{(a^4q^{2n}, -a^2q^{1/2-n}; q)_\infty}{(a^4, -a^2q^{n+1/2}; q)_\infty} \\ \times {}_3\phi_2 \left[\begin{matrix} q^{-2n}, & -q^{1/2-n}e^{i\theta}/a, & -q^{1/2-n}e^{-i\theta}/a \\ & -aq^{1/2-n}e^{-i\theta}, & -aq^{1/2-n}e^{i\theta} \end{matrix} ; q, a^4q^{2n} \right].$$

By Sears' transformation formula [20, II(a), p. 173],

$$(3.4) \quad {}_3\phi_2 \left[\begin{matrix} q^{-2n}, & -q^{1/2-n}e^{i\theta}/a, & -q^{1/2-n}e^{-i\theta}/a \\ & -aq^{1/2-n}e^{-i\theta}, & -aq^{1/2-n}e^{i\theta} \end{matrix} ; q, a^4q^{2n} \right] \\ = \frac{(a^2; q)_{2n}}{(-aq^{1/2-n}e^{-i\theta}; q)_{2n}} {}_3\phi_2 \left[\begin{matrix} q^{-2n}, & a^2, & -q^{1/2-n}e^{-i\theta}/a \\ & q^{1-2n}/a^2, & -aq^{1/2-n}e^{i\theta} \end{matrix} ; q, q \right]$$

and hence by combining (3.1), (3.3) and (3.4) we find that

$$(3.5) \quad p_n(\cos 2\theta; a, aq^{1/2}, -a, -q^{1/2}/a|q) = \frac{(q; q)_{2n} (-a^2q^{1/2}; q)_n}{(a^4; q)_{2n} (-q^{1/2-n}; q)_n} G_{2n}(\cos \theta; a^2|q),$$

where $G_n(x; \beta|q)$ is as defined in (1.25). This is a q -analogue of the quadratic transformation formula [8, 10.9 (21)]:

$$(3.6) \quad P_n^{(\lambda-1/2, -1/2)}(\cos 2\theta) = \frac{(1/2)_n}{(\lambda)_n} C_{2n}^\lambda(\cos \theta).$$

Similarly one can show that

$$(3.7) \quad \begin{aligned} & p_n(\cos 2\theta; a, aq^{1/2}, -aq^{1/2}, -q/a|q) \\ &= \frac{(q; q)_{2n+1} (-qa^2; q)_n}{(a^4; q)_{2n+1} (-q^{-n}; q)_n} \frac{1+a^2}{e^{i\theta} + ae^{-i\theta}} G_{2n+1}(\cos \theta; a^2|q) \end{aligned}$$

which is a q -analogue of [8, 10.9 (22)]:

$$(3.8) \quad \cos \theta P_n^{(\lambda-1/2, 1/2)}(\cos 2\theta) = \frac{(1/2)_{n+1}}{(\lambda)_{n+1}} C_{2n+1}^\lambda(\cos \theta).$$

It is clear from (3.5) and (3.7) that $G_n(x; q^\lambda|q)$ behaves like the ultraspherical polynomials $C_n^\lambda(x)$ in the limit $q \rightarrow 1$, but otherwise quite different from and not nearly as nice as Rogers' continuous q -ultraspherical polynomials

$$(3.9) \quad C_n(\cos \theta; \beta|q) = \sum_{k=0}^n \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \cos(n - 2k)\theta.$$

The orthogonality and other properties of $C_n(x; \beta|q)$ are by now well known, mainly through the works of Askey and Ismail [4, 5] and most of these results have proved very useful. We do not expect the functions $G_n(x; \beta|q)$ to be half as useful. But the fact that they occur in a q -analogue of an important formula, namely (1.4), for ultraspherical polynomials, is probably reason enough to look at them a bit more closely, study their properties and look for other applications. This we plan to do in a later report.

4. A q -analogue of the Feldheim-Vilenkin formula. From (1.22) we find that

$$(4.1) \quad \begin{aligned} & \int_0^{\pi/2} \rho(\cos 2\phi; a, b, c, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(abc\mu e^{2i\theta}, abc\mu e^{-2i\theta}; q)_n}{(abc\mu^2 e^{2i\phi}, abc\mu^2 e^{-2i\phi}; q)_n} \\ & \cdot p_n(\cos 2\phi; a, b, c, d|q) d\phi \\ &= \frac{1}{2} g(a, b, c, \mu e^{2i\theta}, \mu e^{-2i\theta}|q) \frac{(bc; q)_n}{(bc\mu^2; q)_n} p_n(\cos 2\theta; a\mu, b\mu, c\mu, d\mu^{-1}|q), \end{aligned}$$

where

$$(4.2) \quad \begin{aligned} & \rho(\cos 2\phi; a_1, a_2, a_3, a_4, a_5|q) \\ &= \frac{h(\cos 2\phi, 1)h(\cos 2\phi, -1)h(\cos 2\phi, q^{1/2})h(\cos 2\phi, -q^{1/2})h(\cos 2\phi, a_1 a_2 a_3 a_4 a_5)}{h(\cos 2\phi, a_1)h(\cos 2\phi, a_2)h(\cos 2\phi, a_3)h(\cos 2\phi, a_4)h(\cos 2\phi, a_5)}. \end{aligned}$$

Using (3.5) we then have

$$\begin{aligned}
 (4.3) \quad & \int_0^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -a, \mu e^{2i\theta}, \mu e^{-2i\theta} | q) \frac{(-q^{1/2} a^3 \mu e^{2i\theta}, -q^{1/2} q^3 \mu e^{-2i\theta}; q)_n}{(-q^{1/2} a^3 \mu^2 e^{2i\phi}, -q^{1/2} a^3 \mu^2 e^{-2i\phi}; q)_n} \\
 & \cdot G_{2n}(\cos \phi; a^2 | q) d\phi \\
 & = \frac{1}{2} g(a, aq^{1/2}, -a, \mu e^{2i\theta}, \mu e^{-2i\theta} | q) \frac{(a^4; q)_{2n}}{(a^4 \mu^4; q)_{2n}} G_{2n}(\cos \theta; a^2 \mu^2 | q).
 \end{aligned}$$

Similarly, the use of (3.7) gives

$$\begin{aligned}
 (4.4) \quad & \int_0^{\pi/2} \rho(\cos 2\phi; a, aq^{1/2}, -aq^{1/2}, \mu e^{2i\theta}, \mu e^{-2i\theta} | q) \frac{(-qa^3 \mu e^{2i\theta}, -qa^3 \mu e^{-2i\theta}; q)_n}{(-qa^3 \mu^2 e^{2i\phi}, -qa^3 \mu^2 e^{-2i\phi}; q)_n} \\
 & \cdot \frac{e^{i\theta} + \mu a e^{-i\theta}}{e^{i\phi} + a e^{-i\phi}} G_{2n+1}(\cos \phi; a^2 | q) d\phi \\
 & = \frac{1}{2} g(a, aq^{1/2}, -aq^{1/2}, \mu e^{2i\theta}, \mu e^{-2i\theta} | q) \frac{1 + a^2 \mu^2}{1 + a^2} \frac{(a^4; q)_{2n+1}}{(a^4 \mu^4; q)_{2n+1}} \\
 & \cdot G_{2n+1}(\cos \theta; a^2 \mu^2 | q).
 \end{aligned}$$

Combining (4.3) and (4.4) we get the formula

$$\begin{aligned}
 (4.5) \quad & \int_0^{\pi/2} K(\cos \theta, \cos \phi; a^2 | q) \frac{h(\cos 2\phi, -a^3 \mu^2 q^{(n+1)/2})}{h(\cos 2\theta, -a^3 \mu q^{(n+1)/2})} \\
 & \cdot \frac{(-aq^{(n+1)/2 - [n/2]} e^{-2i\phi}, -aq^{(1-n)/2 + [n/2]} e^{2i\phi}; q)_\infty}{(-a\mu q^{(n+1)/2 - [n/2]} e^{-2i\theta}, -a\mu q^{(1-n)/2 + [n/2]} e^{2i\theta}; q)_\infty} \\
 & \cdot e^{2i(\theta - \phi)(n/2 - [n/2])} \cdot G_n(\cos \phi; a^2 | q) d\phi \\
 & = \frac{(a^4; q)_n}{(a^4 \mu^4; q)_n} G_n(\cos \theta; a^2 \mu^2 | q),
 \end{aligned}$$

where

$$\begin{aligned}
 (4.6) \quad & K(\cos \theta, \cos \phi; a^2 | q) = \frac{(q, a^4, \mu^2, a^2 \mu^2; q)_\infty}{\pi (a^4 \mu^4, a^2; q)_\infty} \\
 & \cdot \frac{h(\cos 4\phi, 1) h(\cos 4\theta, a^2 \mu^2)}{h(\cos 4\phi, a^2) h(\cos 2\phi, \mu e^{2i\theta}) h(\cos 2\phi, \mu e^{-2i\theta})}, \quad |a| < 1, 0 < \mu < 1.
 \end{aligned}$$

Replacing a, μ by $q^{\lambda/2}, q^{(\nu-\lambda)/2}$, $\nu > \lambda > -1/2$, it can be shown by using the q -gamma function

$$(4.7) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad \Gamma(x) = \lim_{q \rightarrow 1} \Gamma_q(x),$$

that (4.5) reduces to

$$\begin{aligned}
 (4.8) \quad & \frac{C_n^\nu(\cos \theta)}{C_n^\nu(1)} \frac{\sin^{2\nu-1} \theta}{\cos^{n+2\lambda+1} \theta} \\
 & = \frac{2\Gamma(\nu + 1/2)}{\Gamma(\lambda + 1/2)\Gamma(\nu - \lambda)} \int_0^\theta \sin^{2\lambda} \phi \frac{[\cos^2 \phi - \cos^2 \theta]^{\nu-\lambda-1}}{\cos^{n+2\nu} \phi} \frac{C_n^\lambda(\cos \phi)}{C_n^\lambda(1)} d\phi
 \end{aligned}$$

which is equivalent to Feldheim-Vilenkin formula (1.4); see Askey [1, pp. 23–24]. The evaluation of the limit of $K(\cos \theta, \cos \phi; |q)$ as $q \rightarrow 1$ is a bit tricky, but similar calculations were done in Nassrallah and Rahman [16].

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DEPARTMENT OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO K1S 5B6, CANADA