PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

CESÀRO AND BOREL-TYPE SUMMABILITY

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(Communicated by R. Daniel Mauldin)

ABSTRACT. Though summability of a series by the Cesàro method C_p does not in general imply its summability by the Borel-type method (B, α, β), it is shown that the implication holds under an additional condition.

1. Introduction. Suppose throughout that $\sum_{n=0}^{\infty} a_n$ is a series with partial sums $s_n := \sum_{k=0}^{n} a_k$, and that $\alpha > 0$ and $\alpha N + \beta > 0$ where N is a nonnegative integer. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (B, α, β) to s if

$$\alpha e^{-x} \sum_{n=N}^{\infty} s_n \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to s \quad \text{as } x \to \infty.$$

The Borel-type summability method (B, α, β) is regular, and $(B, 1, 1)$ with $N = 0$ is the standard Borel summability method B.

We shall also be concerned with the Cesàro summability method C_p $(p > -1)$ and the Valiron method V_α defined as follows:

$$\sum_{n=0}^{\infty} a_n = s(C_p) \quad \text{if } c_n^p := \frac{s_n^p}{\binom{n+p}{n}} \to s \quad \text{as } n \to \infty$$

where

$$s_n^p := \sum_{k=0}^{n} \binom{n-k+p-1}{n-k} s_k;$$

$$\sum_{n=0}^{\infty} a_n = s(V_\alpha) \quad \text{if } \left(\frac{\alpha}{2\pi n}\right)^{1/2} \sum_{k=0}^{\infty} \exp\left(-\frac{\alpha(n-k)^2}{2n}\right) s_k \to s \quad \text{as } n \to \infty.$$

Consider the series $\sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty} n^{a-1} \exp(Ain^a)$ where $A > 0$ and $0 < a < 1/2$. It is known [**5**, p. 213] that this series is summable C_p for every $p > 0$ but is not convergent. However, since $a_n = o(n^{-1/2})$, it follows by the Borwein Tauberian Theorem [**1**, Theorem 1] that the series is not summable (B, α, β) for any α and β. This example shows that, in general, summability C_p does not imply summability (B, α, β). The following theorem indicates how to strengthen the C_p summability hypothesis in order to ensure summability (B, α, β).

THEOREM 1. *Suppose that p is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$.*

The special case $\alpha = \beta = 1$, $p = 1$ of Theorem 1 has been proved by Hardy [**5**, Theorem 149]. Hardy and Littlewood [**4**, §3] proved that the condition

Received by the editors July 14, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 40G05, 40G10.
This research was supported in part by the Natural Sciences and Engineering Council of Canada.

$c_n^p = s+o(n^{-1/2})$ is not sufficient for the summability of $\sum a_n$ by the Borel method. Hyslop [**7**, Theorem VIII] has obtained a more general result than Hardy, namely the case $\alpha = \beta = 1$ of Theorem 1. More recently, Swaminathan [**10**] has proved Theorem 1 with $p = 1$ and (B, α, β) summability replaced by the more general $F(a, q)$ summability introduced by Meir [**9**].

2. Preliminary results.

LEMMA 1 [**8**, LEMMA 7]. *Let $m < x_0 < n-1$ where m, n are integers and let the nonnegative function $f(x)$ be increasing on $[m, x_0]$ and decreasing on $[x_0, n]$. Then*

$$\sum_{k=m}^{n} f(k) \le \int_m^n f(x)dx + f(x_0).$$

LEMMA 2 [**2**, THEOREM 3]. *Suppose that $s_n = O(n^r)$ where $r \ge 0$. Then $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$ if and only if $\sum_{n=0}^{\infty} a_n = s(V_\alpha)$.*

THEOREM 2 (CF. [**6**, THEOREM 2]). *Suppose that p is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n = s(V_\alpha)$.*

PROOF. Suppose, as we may without loss of generality, that $s = 0$.

Let $v_n(x) := \exp(-\alpha(n-x)^2/2n)$ and denote the pth difference of $v_n(k)$ by $\Delta^p v_n(k)$, so that

$$\Delta^p v_n(k) = \sum_{r=0}^{p} \binom{p}{r} (-1)^r v_n(k+r).$$

Applying Abel's partial summation formula p ($< m$) times, we have that

$$\sum_{k=0}^{m} s_k v_n(k) = \sum_{k=0}^{m-p} s_k^p \Delta^p v_n(k) + \sum_{r=0}^{p-1} s_{m-r}^{r+1} \Delta^r v_n(m-r).$$

Letting $m \to \infty$ and applying the limitation theorem for Cesàro summability [**5**, Theorem 46], we see that

$$F(n) := \sum_{k=0}^{\infty} s_k v_n(k) = \sum_{k=0}^{\infty} s_k^p \Delta^p v_n(k).$$

In order to prove the theorem we must show that $F(n) = o(n^{1/2})$. Since, by the hypothesis, $s_k^p = o(k^{p/2})$ as $k \to \infty$ and $k^{p/2}\Delta^p v_n(k) = o(n^{1/2})$ as $n \to \infty$, it suffices to show that

$$G(n) := n^{-1/2} \sum_{k=0}^{\infty} k^{p/2} |\Delta^p v_n(k)| \tag{1}$$

is bounded.

It is familiar that $\Delta^p v_n(k) = (-1)^p v_n^{(p)}(k+c)$ for some $c \in [0, p]$. Hence there is a $\theta = \theta(n, k) \in [0, p]$ such that

$$|\Delta^p v_n(k)| \le |v_n^{(p)}(k+\theta)|. \tag{2}$$

Since $v_n^{(p)}(x) = v_n(x)\sum_{0\le r\le p/2} b_r(n-x)^{p-2r}n^{r-p}$, where the b_r's are constants, we get from (1) and (2) that

$$G(n) = O\left(\sum_{0\le r\le p/2} |b_r| n^{r-p-1/2} \sum_{k=0}^{\infty} k^{p/2}|n-k-\theta|^{p-2r} v_n(k+\theta)\right).$$

Therefore to establish that $G(n)$ is bounded it is enough to show that, for $0 \le r \le p/2$ and $0 \le \theta \le p$,

$$H(n) := \sum_{k=0}^{\infty} k^{p/2}|n-k-\theta|^{p-2r} v_n(k+\theta) = O(n^{p-r+1/2}).$$

Write

$$\text{(3)}\qquad H(n) = \left\{\sum_{k=0}^{n-p-1} + \sum_{k=n-p}^{n} + \sum_{k=n+1}^{\infty}\right\} k^{p/2}|n-k-\theta|^{p-2r} v_n(k+\theta) := \sum_1 + \sum_2 + \sum_3 .$$

Since $|n-k-\theta| \le 2p$ for $0 \le \theta \le p$ and $n-p \le k \le n$, and $0 < v_n(k+\theta) \le 1$, it is immediate that

$$\text{(4)}\qquad \sum_2 = O(n^{p/2}).$$

Next, setting $f(x) := x^{p-2r}\exp(-\alpha x^2/2n)$ and applying Lemma 1, we have that

$$\begin{aligned}\sum_1 &\le \sum_{k=0}^{n-p-1} k^{p/2}(n-k)^{p-2r}v_n(k+p) \le \sum_{k=p}^{n-1} k^{p/2}(n-k+p)^{p-2r}v_n(k)\\ &\le Mn^{p/2}\sum_{k=p}^{n-1} f(n-k) \le Mn^{p/2}\sum_{k=1}^{n} f(k)\\ &\le Mn^{p/2}\int_1^n f(x)dx + MCn^{p/2}\left(\frac{(p-2r)n}{\alpha}\right)^{p/2-r},\end{aligned}$$

where $M := (1+p)^{p-2r}$ and $C := \exp(r-p/2)$. Letting $u = \alpha x^2/2n$, we get that

$$\text{(5)}\qquad \sum_1 = O\left(n^{p-r+1/2}\int_0^{\infty} u^{(p-1)/2-r}e^{-u}du\right) + O(n^{p-r}) = O(n^{p-r+1/2}).$$

Further, with M and $f(x)$ as above and $g(x) := x^{3p/2-2r}\exp(-\alpha x^2/2n)$, we see that

$$\begin{aligned}\sum_3 &\le \sum_{k=n+1}^{\infty} k^{p/2}(k-n+p)^{p-2r}v_n(k)\\ &\le M\left(\sum_{k=n+1}^{2n} + \sum_{k=2n+1}^{\infty}\right) k^{p/2}(k-n)^{p-2r}v_n(k)\\ &\le M(2n)^{p/2}\sum_{k=1}^{n} f(k) + M2^{p/2}\sum_{k=n+1}^{\infty} g(k) := \sum_{3,1} + \sum_{3,2} .\end{aligned}$$

As above $\sum_{3,1} = O(n^{p-r+1/2})$. And finally, as $n \to \infty$,

$$\begin{aligned}\sum_{3,2} &= O\left(\int_n^\infty g(x)dx\right) + o(1) \\ &= O\left(n^{3p/4-r+1/2}\int_{\alpha n/2}^\infty u^{3p/4-r-1/2}e^{-u}\,du\right) + o(1) \\ &= o(1).\end{aligned}$$

Thus,

$$\sum_3 = O(n^{p-r+1/2}) + o(1) \quad \text{as } n \to \infty. \tag{6}$$

It now follows from (3)–(6) that $H(n) = O(n^{p-r+1/2})$. This completes the proof. □

3. Proof of Theorem 1. The limitation theorem for Cesàro summability [**5**, Theorem 46] implies that $s_n = o(n^p)$. Therefore, by Theorem 2 and Lemma 2, we have that $\sum_{n=0}^\infty a_n = s(B, \alpha, \beta)$. □

4. Related results. The methods of Euler E_δ, Meyer-Konïg S_δ, and Taylor T_δ $(0 < \delta < 1)$ are defined as follows:

$$\sum_{n=0}^\infty a_n = s(E_\delta) \quad \text{if } \sum_{k=0}^n \binom{n}{k}\delta^k(1-\delta)^{n-k}s_k \to s \quad \text{as } n \to \infty;$$

$$\sum_{n=0}^\infty a_n = s(S_\delta) \quad \text{if } (1-\delta)^{n+1}\sum_{k=0}^\infty \binom{n+k}{k}\delta^k s_k \to s \quad \text{as } n \to \infty;$$

$$\sum_{n=0}^\infty a_n = s(T_\delta) \quad \text{if } (1-\delta)^{n+1}\sum_{k=0}^\infty \binom{n+k}{k}\delta^k s_{n+k} \to s \quad \text{as } n \to \infty.$$

These methods, as well as the Borel-type and Valiron methods, are contained in the $F(a,q)$ family of methods mentioned in the introduction. The following theorem generalizes Swaminathan's result [**10**], via Theorem 2 and [**3**, Satz III], for the Euler, Meyer-Konïg, and Taylor methods.

THEOREM 3. *Suppose that p is a nonnegative integer and that $c_n^p = s+o(n^{-p/2})$ as $n \to \infty$. Then for $0 < \delta < 1$, the series $\sum_{n=0}^\infty a_n$ is summable to s by the E_δ, S_δ, and T_δ methods.*

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