INVARIANT LAGRANGIAN SUBSPACES

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ABSTRACT. It is proved that on Hilbert spaces with strong symplectic form, every symplectic operator $I + C$ with C compact has an invariant Lagrangian subspace.

1. Introduction. It is a well-known fact that every symplectic operator on a finite-dimensional complex vector space has a Lagrangian invariant subspace. In this note we generalize the result on invariant Lagrangian subspaces to symplectic operators of the form $I + C$ with C compact on a Hilbert space with strong symplectic form. The proof uses a method due to Arveson and Feldman [1].

The relation between Lagrangian subspaces and complex structures is of importance in the quantization problem [8]. The result on existence of invariant Lagrangian subspaces proved here is a step towards extending the results of Paneitz [4] which assume a bound on C.

Let $(H, \langle\,,\,\rangle)$ be a separable Hilbert space over the field $\mathbb{C}$ equipped with a strong symplectic form ω, i.e. a skew-symmetric continuous bilinear form such that the induced mapping $\hat{\omega}\colon H \to H^*$ defined by $\langle\hat{\omega}x, y\rangle = \omega(x, y)$ is an isomorphism.

Let $L(H)$ denote the space of continuous linear automorphisms of H. An element $A \in L(H)$ is said to be symplectic if $\omega(Ax, Ay) = \omega(x, y)$, $\forall x, y \in H$ and the subgroup of $\mathrm{GL}(H)$ consisting of symplectic operators with respect to ω is denoted by $\mathrm{Sp}(H, \omega)$. The subgroup of $\mathrm{Sp}(H, \omega)$ consisting of operators of the form $(I + C)$ with C compact is denoted by $\mathrm{Sp}_C(H, \omega)$ and the corresponding Lie algebra by $\mathbf{sp}_C(H, \omega)$.

Let $L \subset H$ be a linear subspace. Then we define the ω-orthogonal complement $L^\perp$ of L by

$$L^\perp = \{x \in H | \omega(x, y) = 0, \ \forall y \in L\}.$$

A subspace $L \subset H$ is said to be isotropic if $L \subset L^\perp$, coisotropic if $L^\perp \subset L$ and symplectic if $L \cap L^\perp = \{0\}$. A closed isotropic subspace L is said to be Lagrangian if there exists another closed isotropic subspace L' such that $H = L \oplus L'$.

The proof of [7, Proposition 5.1] extends easily to Hilbert spaces over $\mathbb{C}$ to show that a maximal isotropic subspace (i.e. $L = L^\perp$) is Lagrangian in the present case.

We are now ready to state our main theorem.

THEOREM 1. *Let (H, ω) be as above. Then, for any $A \in \mathrm{Sp}_C(H, \omega)$, there exists a Lagrangian subspace $L \subset H$ which is invariant under A, i.e. $AL \subset L$. The same statement holds for $X \in \mathbf{sp}_C(H, \omega)$.*

Theorem 1 will be proved in §2. In the corollaries below we record some facts that can be proved using Theorem 1. It is also possible to prove these directly using finite-dimensional approximations of C and the continuity of spectral data.

Let C be a compact operator on H. Then the spectrum of C consists of an at most countable number of points with 0 as their only point of accumulation. The nonzero points in the spectrum are always eigenvalues of finite multiplicity, but the point 0 which is always in the spectrum can be "infinitely degenerate" and indeed, a compact operator may lack eigenvalues. See [5] for discussion and examples. To study the properties of the spectrum of symplectic operators in Sp_C we use the following normal form, which is a standard result on symplectic operators with an invariant Lagrangian subspace; see [6, Proposition 2.5] for the real case.

PROPOSITION 1. *Let (H, ω) be as above, let $A \in \mathrm{Sp}(H, \omega)$ and assume that A has an invariant Lagrangian subspace L. Then there is a symplectomorphism $\rho: (H, \omega) \rightarrow (L \oplus L^*, \Omega)$, where Ω is the canonical symplectic form on $L \oplus L^*$, which transforms A to the form*

$$\rho \circ A \circ \rho^{-1} = \begin{bmatrix} B & BE \\ 0 & (B^*)^{-1} \end{bmatrix},$$

where $B: L \rightarrow L$ is invertible and $E: L^* \rightarrow L$ defines a symmetric bilinear form on L^*, which in this case is bounded. For $X \in \mathbf{sp}(H, \omega)$ the corresponding form is

$$\rho \circ X \circ \rho^{-1} = \begin{bmatrix} Z & E \\ 0 & -Z^* \end{bmatrix},$$

where Z is any bounded operator and E is as above.

This normal form immediately gives the following characterization of the spectrum of symplectic operators in Sp_C.

COROLLARY 1. *Let (H, ω) be a symplectic Hilbert space over $\mathbf{R}$ with strong symplectic form ω and let $A \in \mathrm{Sp}_C(H, \omega)$. Let λ be in the spectrum of A. Then $\lambda, \bar{\lambda}, \lambda^{-1}\bar{\lambda}^{-1}$ all have the same multiplicity. In particular, the multiplicity of 1 is even or infinite.*

For $X \in \mathrm{sp}_C(H, \omega)$, the corresponding statement is that if λ is in the spectrum of X, then $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ all have the same multiplicity.

PROOF. Extend A and ω to the complex Hilbert space $H \otimes \mathbb{C}$. Then Theorem 1 gives an invariant Lagrangian subspace so we can apply Proposition 1. The result now follows from standard spectral theory. □

Let $\mathrm{Sp}_r(H, \omega)$ (with Lie-algebra $\mathbf{sp}_r(H, \omega)$) denote the space of symplectic operators of the form $A = I + C$ with $C \in L^r(H)$, where, for $r = 1$, L^1 denotes the space of traceclass operators on H and for $r = 0$, L^0 denotes the space of operators on H with finite rank.

For an operator of the form $A = I + C$ with $C \in L^1(H)$,

$$\det(A) = \prod_{i=1}^{N(A)} \lambda_i,$$

where $\det(A)$ denotes the Fredholm determinant of A, $\lambda_i(A)$ are its eigenvalues, with multiplicity $N(A)$ (see [5]). Similarly, for $X \in L^1(H)$,

$$\operatorname{tr}(X) = \sum_{i=1}^{N(X)} \lambda_i(X).$$

Applying this to the result in Corollary 1 gives

COROLLARY 2. *Let $A \in \operatorname{Sp}_1(H, \omega)$. Then $\det(A) = 1$. Let $X \in \mathbf{sp}_1(H, \omega)$. Then $\operatorname{tr}(X) = 0$.*

2. Proof of Theorem 1. We will need a few preliminaries.

LEMMA 1. *Let $A \in \operatorname{Sp}_C(H, \omega)$ and let $L \subset H$ be a closed invariant subspace of A. Then $L^\perp$ is a closed invariant subspace of A.*

PROOF. Since $A \in \operatorname{Sp}_C$, $A|_L : L \to L$ is injective and has index zero and hence is surjective. Thus, every $y \in L$ satisfies $y = Ax$ for some $x \in L$ which implies $A^{-1}y = A^{-1}Ax = x \in L$. This shows that $A^{-1}L = L$. Let now $x \in L^\perp$ and note that $\omega(Ax, y) = \omega(x, A^{-1}y) = 0$ for all $y \in L$. It follows by the definition of $L^\perp$ that $Ax \in L^\perp$. $\square$

LEMMA 2. *Let $L \subset H$ be an isotropic subspace. Then $L^\perp / L$ has an induced symplectic form ω'. Let $W \subset L^\perp / L$ be an isotropic subspace with respect to ω'. Then there is a unique subspace W_0 of $L^\perp$ such that $W_0 / L = W$. Further, W_0 is isotropic.*

PROOF. This is standard, except for the last statement. Let W_0 be the (unique) preimage of W under the projection $\pi : L^\perp \to L^\perp / L$. Then, for $x, y \in W^0$ we have $\omega(x, y) = \omega'(\pi(x), \pi(y)) = 0$. $\square$

LEMMA 3. *Let $A \in \operatorname{Sp}_0(H, \omega)$. Then A has an invariant Lagrangian subspace.*

REMARK. The proof of Lemma 3 is a straightforward extension of the proof for the finite-dimensional case (see [2]) but is included since it is used as a model for the final proof.

PROOF. We can use the same method as in the finite-dimensional case, since $A \in \operatorname{Sp}_0(H, \omega)$ implies that A has a complete set of generalized eigenvectors. Thus, let $e \in H$ be an eigenvector of A and let L denote the span of e. Then L is an invariant isotropic subspace for A.

Consider the set of all nontrivial invariant isotropic subspaces for A. By Zorn's Lemma, this has a maximal element which we denote by L. We claim that L is also maximal isotropic in H and hence Lagrangian by [7, Proposition 5.1].

To prove this, we assume the contrary, i.e. that L is not maximal isotropic in H. In particular, this means that $L^\perp / L$ is nontrivial. Note that $L^\perp$ is also invariant by Lemma 1 so we get an induced operator $A_{L^\perp/L}$ on $L^\perp/L$. Clearly, $A_{L^\perp/L} \in \operatorname{Sp}_0(L^\perp/L, \omega_{L^\perp/L})$ so we again find an eigenvector which spans a one dimensional isotropic subspace $W \subset L^\perp/L$. Thus, we can apply Lemma 2 to get an invariant isotropic subspace $L' \subset H$ such that $L \subset L'$ with strict inclusion. Hence, L is not maximal among the invariant isotropic subspaces for A, which is absurd.

Consider the effect of these operations on P and $\phi(P)$. By the construction of ρ, it takes $L \times H/L \to L \times \hat{\omega}(H/L) = L \times L^*$, so we see that $\rho_* P$ becomes projection onto the first component and $\rho_* \phi(P)$ becomes projection onto the second component.

By the linearity of ρ_* and the isometry properties of ρ, we can easily deduce the statements in the Lemma. For part (1), note that $\phi(P)$ is an orthogonal projection since $\rho_* \phi(P)$ is one and its range is $\rho^{-1}(0 \times L^*)$ which is the image of a Lagrangian subspace by a symplectomorphism and hence Lagrangian.

Part (2) follows from the selfadjointness of P and $\phi(P)$ and the fact that $0 = (\rho_* P)(\rho_* \phi(P)) = \rho_*(P\phi(P))$.

The statement in part (3) follows from the fact that $\rho_*(P + \phi(P)) = I_V$, the identity operator on V. □

LEMMA 5. *Let (H, ω) and W be as in Lemma 4. Let $B \in L^1(W)$ be injective and nonnegative and assume that $\operatorname{tr}(B) = 1/2$. Let the state $\Omega \colon L(H) \to \mathbf{C}$ be given by $\Omega(X) = \operatorname{tr}((B \oplus B^*) \circ X)$. Then Ω is a faithful normal state on $L(H)$ and $\Omega(P) = 1/2$ for any orthogonal projection P onto a Lagrangian subspace of (H, ω).*

PROOF. The only statement that is not obvious is that $\Omega(P) = 1/2$. To see this, note that by Lemma 4,

$$1 = \Omega(I) = \Omega(P) + \Omega(\phi(P)).$$

Now, using the matrix form of $\hat{\omega}$ and $\hat{\omega}^{-1} = \hat{\omega}^*$, we get

$$\phi(B \oplus B^*) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} B^* & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B^* \end{bmatrix} = B \oplus B^*.$$

Thus we have $\Omega(\phi(P)) = \operatorname{tr} \phi((B \oplus B^*) \circ P)$. By the isometry properties of $\hat{\omega}$ we find that $\Omega(\phi(P)) = \Omega(P)$, so we get $2\Omega(P) = 1$. □

We now apply the above and the method of proof used in [1] to the problem of finding an invariant isotropic subspace for operators in $\operatorname{Sp}_C(H, \omega)$. This will allow us to generalize the method used in the proof of Lemma 3 to operators in Sp_C. By the remark before Lemma 4 we can, without loss of generality assume that $H = W \times W^*$ and that ω is the canonical symplectic form on H.

LEMMA 6. *Let (H, ω) be as above and let $A \in \operatorname{Sp}_C(H, \omega)$. Then A has a nontrivial invariant isotropic subspace.*

PROOF. If A has an eigenvector e, then the span of e is an invariant isotropic subspace. Thus, in the following we may without loss of generality assume that A has no eigenvectors.

It is well known that $\operatorname{Sp}_C(H, \omega)$ is the closure with respect to the norm topology of $\operatorname{Sp}_0(H, \omega)$ (to see this, use the surjectivity of the exponential mapping and apply the approximation property for compact operators in $\mathbf{sp}_C(H, \omega)$, cf. [3]). Thus, let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\operatorname{Sp}_0(H, \omega)$ such that $\|A_n - A\|_{L(H)} \to 0$ as $n \to \infty$. Let L_n be an invariant Lagrangian subspace for A_n, given by Lemma 3 and let P_n be the orthogonal projection onto L_n. Let Ω be as in Lemma 5, so that $\Omega(P_n) \equiv 1/2$.

Now let $P \in L(H)$ be an accumulation point of $\{P_n\}$ with respect to the weak topology and in the following, let $\{P_n\}$ denote a convergent subsequence. Then $\Omega(P) = 1/2$, so $P \neq 0$ and if we let $M = \{x | Px = x\}$, then $M \neq H$.

By construction, $(P_n - I)A_n P_n = 0$ for all n, so

$$(2.2) \quad \|(P_n - I)AP_n\| \leq \|(P_n - I)(A - A_n)P_n\| + \|(P_n - I)A_n P_n\| \leq \|A - A_n\|,$$

which tends to 0 as $n \to \infty$. This is the crucial property used in [1]. Following the proof in [1] we find that for $x \in M$, $P_n x \to x$ in norm, since

$$(2.3) \quad \|P_n x - x\|^2 = (P_n x, P_n x) - 2(P_n x, x) + (x, x) = -(P_n x, x) + (x, x) \to 0,$$

where we have used the projection property and selfadjointness of P_n It follows that

$$\omega(x, y) = \lim_{n \to \infty} \omega(P_n x, P_n y) = 0, \quad \forall x, y \in M,$$

so M is an isotropic subspace of H. To see that M is also invariant, note that for $x \in M$, $AP_n x \to Ax$ in norm by (2.3), so $P_n AP_n x \to PAx$ weakly. Now, using (2.2) we can conclude that $AP_n x \to PAx$ weakly, but by the above, $AP_n x \to Ax$ in norm, so $Ax = PAx$ and hence $Ax \in M$.

Finally, we have to check that M is nontrivial. Let $C = A - I$. Then, since $A \in \mathrm{Sp}_C(H, \omega)$ by assumption, C is compact. Let $x \in H$ be arbitrary. By the weak convergence of $P_n \to P$ and the compactness of C it follows that $CP_n x \to CPx$ in norm. Again by the weak convergence of P_n to P, we have that $P_n CP_n x \to PCPx$ weakly, but using (2.2) we find that $\|P_n CP_n - CP_n\| \to 0$, so in fact $CP_n x \to PCPx$ weakly. But we just proved that the left side converges in norm to CPx and it follows that $CPx = PCPx$.

This completes the proof since by the remark at the beginning, we can assume that A has no eigenvalues and in particular that the nullspace of C is trivial. Hence by choosing $x \in H$ so that $Px \neq 0$, which is always possible since $P \neq 0$, CPx provides a nonzero element of M. Thus, M is a nontrivial isotropic subspace, invariant under A. □

REMARK. Apart from using the projections P_n onto Lagrangian subspaces, the special state Ω and noting that M is isotropic, the proof of Lemma 6 follows very closely that of [1, Theorem, p. 62]. The author is grateful to Professor Arveson for pointing this paper out to him. □

We are now ready to complete the proof of Theorem 1. Let $A \in \mathrm{Sp}_C(H, \omega)$. By Lemma 6, there is a nontrivial invariant isotropic subspace for A. Consider as in the proof of Lemma 3, the set of all invariant isotropic subspaces for A. By Zorn's Lemma, this set has a maximal element which we call L. We wish to show that L is a maximal isotropic subspace of H and hence Lagrangian.

To do this, assume that L is not maximally isotropic in H. Then $L^\perp / L$ is a nontrivial symplectic space and arguing as in the proof of Lemma 3, we see that the induced operator $A_{L^\perp / L}$ is in $\mathrm{Sp}_C(L^\perp / L, \omega_{L^\perp / L})$ and hence by Lemma 6 that there is a nontrivial invariant isotropic subspace for $A_{L^\perp / L}$. By lifting to H as in the proof of Lemma 3, we find that L is not maximal among the invariant isotropic subspaces for A, a contradiction. It follows that L is maximally isotropic in H. The same method gives the result for $X \in \mathbf{sp}_C(H, \omega)$. This completes the proof of Theorem 1.

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