RANDOM APPROXIMATIONS AND RANDOM FIXED POINT THEOREMS FOR NON-SELF-MAPS

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ABSTRACT. Recently, Sehgal and Singh [18] and Papageorgiou [16] considered different random versions of a very interesting theorem of Fan [4]. Instead of compact convex domain, here we consider a continuous condensing or nonexpansive random map defined on a closed ball (or closed convex set with bounded range). We prove it is true for certain spaces. As applications of our theorems, some random fixed point theorems of non-self-maps are derived.

1. Introduction and preliminaries. Random fixed point theory has just received much attention for the last decade, since the publication of the paper by Bharucha-Reid [1] in 1976, of which the stochastic version of the well-known Schauder's fixed point theorem was proved. In this paper, we will consider a stochastic version of a very interesting theorem of Fan [4, Theorem 2] which is stated as follows.

Let K be a nonempty compact convex set in a normed linear space X. For any continuous map f from K into X, there exists a point u in K such that $\|u - f(u)\| = d(f(u), K)$.

Various aspects (nonstochastic) of this theorem has been studied by Fan [5], Ha [6], Lin [11–14], and Reich [17]. Recently, Sehgal and Singh [18], and Papageorgiou [16] gave stochastic versions of the above theorem. Here, instead of considering a continuous map defined on a compact convex set as in [18], we consider a continuous condensing or a nonexpansive map defined on a closed convex set with bounded range (or a closed ball). Roughly speaking in this paper, we will prove that the random version of above theorem is true for a continuous condensing random (for definitions see below) operator $f: \Omega \times S \to X$ where S is a closed ball in a separable Banach space X and Ω is a measurable space. We will also prove that the random version of above theorem is true for a continuous condensing (or nonexpansive) random operator $f: \Omega \times S \to X$ where S is a closed convex subset of a separable Hilbert space and $f(\omega, S)$ is bounded for $\omega \in \Omega$. As applications of our theorems, some stochastic fixed point theorems are derived. We also remark that a related (but different) result has been obtained by Sehgal and Waters [19].

Now, we state some definitions.

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω. Let X be a Banach space, a map $f: \Omega \to X$ is called measurable if for each open set B of

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X, $f^{-1}(B) \in \Sigma$. If f is a multifunction, then $f^{-1}(B) = \{\omega | f(\omega) \cap B \neq \varnothing\}$. Let S be a nonempty subset of X, and a map $f \colon \Omega \times S \to X$ is called a random operator if for each fixed $x \in S$, the map $f(\cdot, x) \colon \Omega \to X$ is measurable. A measurable map $\varphi \colon \Omega \to S$ is a random fixed point of the random operator f if $f(\omega, \varphi(\omega)) = \varphi(\omega)$, for each $\omega \in \Omega$.

Let B be a nonempty bounded subset of S, and let $\alpha(B) = \inf\{c > 0 | B$ can be covered by a finite number of sets of diameter $\leq c\}$. A map $f \colon S \to X$ is called condensing (e.g. see [15]) if for each bounded subset B of S with $\alpha(B) > 0$, we have $\alpha(f(B)) < \alpha(B)$; f is nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for $x, y \in S$; f is compact if $f(S)$ is totally bounded.

It is easy to see (e.g. see [15]) that (i) $\alpha(A) \leq \alpha(B)$ if $A \subset B$, (ii) $\alpha(\mathrm{co}\, A) = \alpha(A)$, (iii) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, (iv) $\alpha(A) = 0$ iff A is totally bounded.

A random operator $f \colon \Omega \times S \to X$ is continuous (condensing, nonexpansive, compact etc.) if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous (condensing, nonexpansive, compact etc.) (e.g. see Itoh [8]); f is a contraction if there exists a map $k \colon \Omega \to [0, 1)$ satisfying $\|f(\omega, x) - f(\omega, y)\| \leq k(\omega)\|x - y\|$ for $x, y \in S$, $\omega \in \Omega$ (see [19]).

2. Main results.

THEOREM 1. *Let S be a closed ball with center at origin and radius r in a separable Banach space X, and $f \colon \Omega \times S \to X$ be a continuous condensing random operator. Then there exists a measurable map $\varphi \colon \Omega \to S$ such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S)$$

for each $\omega \in \Omega$.

PROOF. Define $R \colon X \to S$ by

$$R(x) = \begin{cases} x & \text{if } \|x\| \leq r, \\ \dfrac{rx}{\|x\|} & \text{if } \|x\| \geq r. \end{cases}$$

From Nussbaum [15, Corollary 1], R is a continuous 1-set-contractive map. Define $F \colon \Omega \times S \to S$ by $F(\omega, x) = R(f(\omega, x))$. It is easy to see that F is a continuous condensing random operator. From Itoh [8, Theorem 2.1], there exists a random fixed point $\varphi \colon \Omega \to S$ of F, i.e. $F(\omega, \varphi(\omega)) = \varphi(\omega)$ for each $\omega \in \Omega$. Now, we will prove that this measurable map φ satisfies the desired property. For each $\omega \in \Omega$,

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = \|F(\omega, \varphi(\omega)) - f(\omega, \varphi(\omega))\|$$
$$= \|R(f(\omega, \varphi(\omega))) - f(\omega, \varphi(\omega))\|$$
$$= \begin{cases} \|f(\omega, \varphi(\omega)) - f(\omega, \varphi(\omega))\| = 0, & \text{if } \|f(\omega, \varphi(\omega))\| \leq r, \\ \left\|\dfrac{rf(\omega, \varphi(\omega))}{\|f(\omega, \varphi(\omega))\|} - f(\omega, \varphi(\omega))\right\| = \|f(\omega, \varphi(\omega))\| - r, \\ \qquad\qquad\qquad\qquad \text{if } \|f(\omega, \varphi(\omega))\| \geq r. \end{cases}$$

For any $y \in S$, we have

$$\|f(\omega, \varphi(\omega))\| - r \leq \|f(\omega, \varphi(\omega))\| - \|y\| \leq \|f(\omega, \varphi(\omega)) - y\|.$$

Therefore

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S). \quad \square$$

THEOREM 2. *Let S be a nonempty separable closed convex subset of a Hilbert space X, and $f: \Omega \times S \to X$ be a continuous condensing random operator such that $f(\omega, S)$ is bounded, for any $\omega \in \Omega$.*

Then there exists a measurable map $\varphi: \Omega \to S$ such that

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S),$$

for each $\omega \in \Omega$.

PROOF. Let p be the proximity map on S, i.e. p is a continuous map from X into S such that for each $y \in X$ we have

$$\|p(y) - y\| = d(y, S).$$

From [3], p is nonexpansive in Hilbert space. Then $p \circ f: \Omega \times S \to S$. It is easy to see that $p \circ f$ is a continuous condensing random operator and $p \circ f(\omega, S)$ is bounded, for any $\omega \in \Omega$. From Itoh [8, Theorem 2.1], there exists a random fixed point of $p \circ f$, i.e. there exists a measurable map $\varphi: \Omega \to S$ such that $p \circ f(\omega, \varphi(\omega)) = \varphi(\omega)$, for each $\omega \in \Omega$. Hence

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = \|p \circ f(\omega, \varphi(\omega)) - f(\omega, \varphi(\omega))\|$$
$$= d(f(\omega, \varphi(\omega)), S), \quad \text{for each } \omega \in \Omega. \quad \square$$

COROLLARY 1 (SEHGAL AND WATERS [19, THEOREM 3]). *Let S be a nonempty separable closed convex subset of a Hilbert space X and $g, h: \Omega \times S \to X$ be random operators such that g is a contraction and h is compact and continuous.*

Then there exists a measurable map $\varphi: \Omega \to S$ such that

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S),$$

where $f = g + h$. If additionally $f(\Omega \times \partial S) \subseteq S$, then φ is a random fixed point.

PROOF. For each fixed $\omega \in \Omega$, and each bounded subset A of S, with $\alpha(A) > 0$, we have

$$\alpha(f(\omega, A)) \leq \alpha(g(\omega, A) + h(\omega, A)) \leq \alpha(g(\omega, A)) + \alpha(h(\omega, A)) = \alpha(g(\omega, A))$$
$$\leq k(\omega)\alpha(A) < \alpha(A).$$

So f is condensing. Hence this corollary follows immediately from Theorem 2 and the Remark after Theorem 4. $\square$

THEOREM 3. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and $f: \Omega \times S \to X$ be a nonexpansive random operator. Then there exists a measurable map $\varphi: \Omega \to S$ such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S),$$

for each $\omega \in S$.

PROOF. Let p be the same as in the proof of Theorem 2. Then $p \circ f: \Omega \times S \to S$ is also a nonexpansive random operator. From Itoh [8, Theorem 2.5], there exists a measurable map $\varphi: \Omega \to S$ such that $p \circ f(\omega, \varphi(\omega)) = \varphi(\omega)$, for each $\omega \in \Omega$. Hence

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S),$$

for each $\omega \in \Omega$. $\square$

We remark that the reason we can have weaker assumption for Theorem 2, namely $f(\omega, S)$ is bounded for $\omega \in \Omega$ (instead of S is bounded in Theorem 3) and S is separable (instead of X is separable in Theorem 3) is that corresponding random fixed point theorem for condensing random self-maps is available (see Itoh [8]). We will consider a weaker assumption for nonexpansive maps after Theorem 6. Certainly, Theorem 1 is still true, if we assume that S is separable (instead of X is separable). (The proof is exactly the same as Theorem 1.) The same remark can be made for the following Theorem 4.

THEOREM 4. *Let S be a closed ball with center at origin and radius r in a separable Banach space X, and $f \colon \Omega \times S \to X$ be a continuous condensing random operator. Moreover, f satisfies one of the following conditions:*

(i) *For each $\omega \in \Omega$, each $x \in S$ with $x \neq f(\omega, x)$, there exists y, depending on ω and x, in $I_S(x) = \{x + c(z - x)|\ \text{some } z \in S, c > 0\}$ such that $\|y - f(\omega, x)\| < \|x - f(\omega, x)\|$.*

(ii) *f is weakly inward (i.e. for each $\omega \in \Omega$, $f(\omega, x) \in \overline{I_S(x)}$ for $x \in S$).*
Then f has a random fixed point.

PROOF. Let f satisfy condition (i). From Theorem 1, there exists a measurable map $\varphi \colon \Omega \to S$ such that

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S)$$

for each $\omega \in \Omega$.

We will prove that φ is the desired random fixed point. If there exists $\omega \in \Omega$ such that $\varphi(\omega) \neq f(\omega, \varphi(\omega))$. From the assumption (i), there exists y in $I_S(\varphi(\omega))$, such that

$$\|y - f(\omega, \varphi(\omega))\| < \|\varphi(\omega) - f(\omega, \varphi(\omega))\|.$$

Since $y \in I_S(\varphi(\omega))$, there exists $z \in S$, $c > 0$ such that $y = \varphi(\omega) + c(z - \varphi(\omega))$. Since $y \notin S$, otherwise contradict the choice of φ, we can assume that $c > 1$. Then $z = y/c + (1 - 1/c)\varphi(\omega) = (1 - \beta)y + \beta\varphi(\omega)$, where $\beta = 1 - 1/c$, $0 < \beta < 1$. Therefore

$$\begin{aligned}\|z - f(\omega, \varphi(\omega))\| &\leq (1 - \beta)\|y - f(\omega, \varphi(\omega))\| + \beta\|\varphi(\omega) - f(\omega, \varphi(\omega))\| \\ &< (1 - \beta)\|\varphi(\omega) - f(\omega, \varphi(\omega))\| + \beta\|\varphi(\omega) - f(\omega, \varphi(\omega))\| \\ &= \|\varphi(\omega) - f(\omega, \varphi(\omega))\|\end{aligned}$$

which contradict the choice of φ. Hence $f(\omega, \varphi(\omega)) = \varphi(\omega)$ for each $\omega \in \Omega$, and φ is the desired random fixed point of f.

If f satisfies condition (ii). For each $\omega \in \Omega$, each $x \in S$ with $x \neq f(\omega, x)$, since $f(\omega, x) \in \overline{I_S(x)}$, there exists y in $I_S(x)$ such that $\|y - f(\omega, x)\| < \|x - f(\omega, x)\|$, and f satisfies condition (i). □

REMARK. If $f(\omega, \partial S) \subset S$ for each $\omega \in \Omega$, then f satisfies condition (ii) of Theorem 4; because $S \subset I_S(x)$, and $I_S(x) = X$ if $x \in \operatorname{int} S$, where ∂S, $\operatorname{int} S$ denote the boundary and interior of S, respectively.

THEOREM 5. *Let S be a nonempty separable closed convex subset of a Hilbert space X, and $f \colon \Omega \times S \to X$ be a continuous condensing random operator such that $f(\omega, S)$ is bounded for any $\omega \in \Omega$. Moreover, f satisfies any one of the conditions (i), (ii) in Theorem 4. Then f has a random fixed point.*

PROOF. The proof is similar to the proof of Theorem 4. Instead of using Theorem 1, we use Theorem 2. □

Similarly, we have the following Theorem 6.

THEOREM 6. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and $f : \Omega \times S \to X$ be a nonexpansive random operator. Moreover f satisfies any one of the conditions* (i), (ii) *in Theorem 4. Then f has a random fixed point.*

For nonexpansive maps, whether we can only assume that $f(\omega, S)$ is bounded for $\omega \in \Omega$ instead of the stronger condition that S is bounded? We will prove that it is true, under the restriction, namely Σ is closed under Suslin operation (for definition of this, see e.g. [20]). We note that there are several different definitions for the measurability of a function $f : \Omega \to X$ in the literature (see [2, 7, 20]); for example, some author called f is measurable if $f^{-1}(B) \in \Sigma$ for every closed subset B of X. However, these different definitions are all equivalent, if Σ is closed under Suslin operation and f is closed valued (e.g. see [20, Theorem 4.2]).

For the following Lemma 1, Theorems $3'$, $6'$, we need assume that Σ is closed under the Suslin operation.

LEMMA 1. *Let S be a nonempty closed convex subset of a separable uniformly convex Banach space X, and $f : \Omega \times S \to S$ be a nonexpansive random operator such that $f(\omega, S)$ is bounded, for any $\omega \in \Omega$. Then f has a random fixed point.*

PROOF. Let $C(\omega) = \overline{\text{co}}(f(\omega, S))$, for $\omega \in \Omega$. Let $D = \{x_n | n \geq 1\}$ be a countable dense subset of S, then it is easy to check that

$$C(\omega) = \overline{\text{co}} \bigcup_{n \geq 1} f(\omega, x_n).$$

From [7, Proposition 2.3 and Theorem 9.1], $C : \Omega \to S$ is a measurable multifunction. Consider the multifunction $L(\cdot)$ defined by $L(\omega) = \{x \in C(\omega) | f(\omega, x) = x\}$. For each $\omega \in \Omega$, $C(\omega)$ is a closed convex bounded subset of S and $f(\omega, \cdot) : C(\omega) \to C(\omega)$, from Kirk [9], $f(\omega, \cdot)$ has a (deterministic) fixed point in $C(\omega)$, i.e. $L(\omega) \neq \varnothing$. It is clear that $L(\cdot)$ is closed valued and $\text{Gr}(L) = \{(\omega, x) \in \Omega \times S | f(\omega, x) - x = 0\} \cap \text{Gr}(C)$, where $\text{Gr}(C) = \{(\omega, x) \in \Omega \times S | x \in C(\omega)\}$.

As we pointed out in the paragraph after Theorem 6, for a closed valued map, all different definition of measurability in the literature are equivalent, if Σ is closed under the Susin operation. So we will not distinguish them for the rest of our proof. Since $(\omega, x) \to f(\omega, x) - x$ is a Carathéodory function (see e.g. [20, p. 863]), from [7, Theorem 6.1] (or see [2, p. 70, Lemma III. 14]), this map is jointly measurable. Since $C(\cdot)$ is measurable, from [20, Theorem 4.2(d)], $\text{Gr}(C) \in \Sigma \times B(X)$, where $B(X) = $ Borel σ-field of X. Thus $\text{Gr}(L) \in \Sigma \times B(X)$. Since $L(\cdot)$ is closed valued, from [20, Theorem 4.2(g)], $L(\cdot)$ is measurable. From the Kuratowski-Ryll Nardzewski selection theorem [10], there exists a measurable function $\varphi : \Omega \to S$ such that $\varphi(\omega) \in L(\omega)$, for each $\omega \in \Omega$. Therefore φ is a random fixed point of f. □

The proof of Lemma 1 is essentially due to the referee, who suggests these arguments to prove the following Theorem $3'$. The author modifies his argument to derive and find our Lemma 1. We note that Itoh [8, Theorem 2.5] proved the above Lemma is true for a weakly compact convex subset S, without assuming that Σ is

closed under Suslin operation. The above lemma is still true if we assume that X is a separable Banach space with normal structure (for definition of this, see Kirk [9]). The proof of this is exactly the same as Lemma 1.

THEOREM 3′. *Let S be a nonempty closed convex subset of a separable Hilbert space X, and $f: \Omega \times S \to X$ be a nonexpansive random operator such that $f(\omega, S)$ is bounded, for any $\omega \in \Omega$. Then there exists a measurable map $\varphi: \Omega \to S$ such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S).$$

PROOF. The proof is exactly the same as Theorem 3, instead of using Itoh [8, Theorem 2.5], we use Lemma 1 to show that $p \circ f$ has a random fixed point.   □

THEOREM 6′. *Let S be a nonempty closed convex subset of a separable Hilbert space X, and $f: \Omega \times S \to X$ be a nonexpansive random operator such that $f(\omega, S)$ is bounded for any $\omega \in \Omega$. Moreover, f satisfies any one of the conditions* (i), (ii) *in Theorem 4. Then f has a random fixed point.*

PROOF. Similar to Theorem 6.   □

Therefore, for the case that Σ is closed under Suslin operation, we have more general Theorem 3′, 6′ (instead of their special case Theorem 3 and 6 respectively).

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