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CHARACTERIZING SHAPE PRESERVING L_1-APPROXIMATION

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ABSTRACT. With certain restrictions, a characterization of the best L_1-approximation to a continuous function from the set of n-convex functions is proved. Under these restrictions the best approximation is shown to be unique. The case $n = 2$ (convex functions) is considered in more detail.

0. Introduction. The term *shape preserving approximation* as used in the title of this paper refers to the approximation of a given function by monotone or convex functions and, in the univariate case, by convex functions of higher-order—the so-called n-convex functions—as well. Our interest is in the univariate case; for results in a multivariate setting see [**3** and **5**].

A real-valued function g defined on a compact real interval $[a, b]$ is called n-*convex* if its nth order divided differences $[x_0, \ldots, x_n]g$ are nonnegative for distinct $x_0, \ldots, x_n$ in $[a, b]$. In particular, a 1-convex function is nondecreasing and a 2-convex function is convex in the usual sense. The set K_n of n-convex functions forms a convex, conical set (a *wedge*), i.e., it is closed under addition and multiplication by a nonnegative constant. Moreover, $K_n \cap (-K_n) = \Pi_{n-1}$, the polynomials of degree at most $n - 1$.

A function $g_0 \in K_n$ is said to be a best L_1-approximation on $[a, b]$ to an integrable function f if

$$\|f - g_0\|_1 := \int_a^b |f - g_0| = \inf\{\|f - g\|_1 : g \in K_n\}.$$

In this paper we consider the problem of characterizing best L_1-approximations to continuous functions by n-convex functions for $n \geq 2$ (for the case $n = 1$ see [**7**]). Existence of shape preserving L_1-approximations in the univariate case has recently been shown in [**4**] and uniqueness was demonstrated in [**6**] for the case $n = 2$ (see also [**8**]). We conjecture that uniqueness holds for $n > 2$ as well; indeed, we prove that uniqueness does hold when certain restrictions are imposed.

We show in Theorem 1 that, with these restrictions, an n-convex function g_0 is a best L_1-approximation to $f \in C[a, b]$ if and only if it is a spline of degree $n - 1$ with simple knots in the zeros of an auxiliary function determined by $\operatorname{sgn}(f - g_0)$, and this function possesses characteristic structural properties.

Our proof is based on a general theorem (due to Rubinštein) characterizing best approximations from wedges [**15**]; the assertions of Theorem 1 are consequences of

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certain special properties of n-convex functions, in particular their integral representation involving a nonnegative Borel measure. In the proof of necessity we have borrowed a technique from Brown [**1**].

In §1 the theoretical foundations needed in the proof of our main result, which appears in §2, are laid. §2 also contains a corollary specializing our results to convex functions ($n = 2$). In this case conditions are given under which a best convex approximation is a linear spline such that, roughly speaking, each linear piece is locally a best convex approximation.

1. Preliminaries. In this section we present some definitions and basic results needed in §2. We start with an overview of some properties of n-convex functions (see, e.g., [**9**, **11**, **13**]). We assume throughout that $n \geq 2$.

An n-convex function on $[a, b]$ may fail to be continuous at most in an endpoint a or b. Consequently, a bounded n-convex function is equivalent in $L_1[a, b]$ to a continuous n-convex function, and hence the assumption of continuity in this case will be tacitly made throughout.

Certain differentiability properties are enjoyed by n-convex functions. In particular, $g^{(n-2)}$ exists and is Lipschitz-continuous on closed subintervals of (a, b), $g_-^{(n-1)}$ exists and is left-continuous and increasing in (a, b) and $g_+^{(n-1)}$ exists and is right-continuous and increasing in (a, b). The set of n-convex functions includes all functions g with n derivatives, such that $g^{(n)} \geq 0$.

DEFINITION 1. *The truncated power $(x - \xi)_+^j$ is defined as $(x - \xi)^j$ if $x \geq \xi$ and as zero otherwise. The expression $(x - \xi)_+^0$ is defined as 1 for $x = \xi$.*

To an n-convex function g on $[a, b]$ we may associate a nonnegative Borel measure μ by setting $\mu([x, y]) := g_+^{(n-1)}(y) - g_-^{(n-1)}(x)$, for $a < x \leq y < b$. If $g_+^{(n-1)}$ and $g_-^{(n-1)}(b)$ are finite then μ may be extended as a bounded measure to all of $[a, b]$ and then g has the representation

$$g(x) = p(x) + \int_a^b \frac{(x - t)_+^{n-1}}{(n - 1)!}\, d\mu(t), \qquad x \in [a, b],$$

where $p \in \Pi_{n-1}$. Otherwise, for each $[\alpha, \beta] \subset (a, b)$ there is a polynomial $p_\alpha \in \Pi_{n-1}$ such that the n-convex function $g_{\alpha,\beta}$ defined on $[a, b]$ by

$$g_{\alpha,\beta}(x) = p_\alpha(x) + \int_\alpha^\beta \frac{(x - t)_+^{n-1}}{(n - 1)!}\, d\mu(t)$$

coincides with g on $[\alpha, \beta]$. Moreover, if g is continuous at the endpoints then the sequence $\{g_{a+1/k, b-1/k}\}_{k=1}^\infty$ converges uniformly to g on $[a, b]$ [**1**, **9**].

DEFINITION 2. *A linearly independent set $\{u_0, \ldots, u_{n-1}\}$ of continuous, real-valued functions defined on $[a, b]$ forms a WT-system if $\det\{u_i(x_j)\}_0^{n-1} \geq 0$ for all $a \leq x_0 < \cdots < x_{n-1} \leq b$. The linear span of a WT-system is called a WT-space.*

For more on WT-spaces see [**14**]. WT-spaces of dimension n are characterized by the property that no element has more than $n - 1$ sign changes. Any basis of a WT-space can be made into a WT-system according to Definition 2 by changing the sign of at most one basis element.

The best known examples of WT-spaces are spaces of spline functions [**14**]:

DEFINITION 3. *The space $S_{n,r} := S_{n,r}(\xi_1, \ldots, \xi_r)$ of spline functions of degree $n - 1$ with r fixed, simple knots $a < \xi_1 < \cdots < \xi_r < b$ is defined as the linear span* *of*

$$\{1, x, \ldots, x^{n-1}, (x-\xi_1)_+^{n-1}, \ldots, (x-\xi_r)_+^{n-1}\}.$$

$S_{n,0}$ *is defined as* Π_{n-1}.

A convenient basis for $S_{n,r}$ is the *B-spline* basis, which may be constructed as follows: We extend $\xi_1, \ldots, \xi_r$ by adding knots

$$\xi_{1-n} < \cdots < \xi_0 = a < \xi_1 < \cdots < \xi_r < b = \xi_{r+1} < \cdots < \xi_{n+r}.$$

The functions $M_1, \ldots, M_{n+r}$, defined as the nth order divided differences

$$M_j(t) := [\xi_{j-n}, \ldots, \xi_j] n(\cdot - t)_+^{n-1},$$

form a basis for $S_{n,r}$ on $[a, b]$ and have the properties

(a) $\operatorname{supp} M_j = (\xi_{n-j}, \xi_j)$ $(j = 1, \ldots, n+r)$, and

(b) $S_{n,r}|_{[\xi_i, \xi_j]} = \operatorname{span}\{M_{i+1}, \ldots, M_{n+j-1}\}$ is a WT-space of dimension $n-1+j-i$, $0 \le i < j \le r+1$.

The following result will be used in the proof of uniqueness (cf. [**10**, Lemma 3]).

PROPOSITION 1. *Let $a := \tau_0 < \tau_1 < \cdots < \tau_N < b := \tau_{N+1}$ be given. Assume that*

$$\sum_{i=0}^{N} (-1)^i \int_{\tau_i}^{\tau_{i+1}} s = 0 \quad \textit{for all } s \in S_{n,r}(\xi_1, \ldots, \xi_r). \tag{1.1}$$

Then, for all $s \in S_{n,r}$,

$$s(\tau_j) = 0 \quad \textit{for all } 1 \le j \le N \Rightarrow s \equiv 0. \tag{1.2}$$

PROOF. We employ [**10**, Lemma 1]. *If U is a k-dimensional WT-space and, for $h \in L_\infty[a, b]$,* $\operatorname{meas}\{h = 0\} = 0$ *and $\int_a^b hu = 0$ for all $u \in U$, then h has at least k sign changes in (a, b).*

In our case, with $h(x) = (-1)^i$ in (τ_i, τ_{i+1}) $(i = 0, \ldots, N)$, (1.1) implies that h has at least $n+r$ sign changes, i.e., $N \ge n+r$. To complete the proof of the lemma, it suffices to find a subset $\{\tau_{i_1}, \ldots, \tau_{i_{n+r}}\}$ of $\{\tau_i\}_{i=1}^N$ such that $\det\{M_i(\tau_{i_j})\}_{i,j=1}^{n+r} \ne 0$. By the well-known Schoenberg-Whitney Theorem [**14**] this is the case precisely when $\tau_{i_j} \in \operatorname{supp} M_j$ $(j = 1, \ldots, n+r)$. Since $\int_a^b hM_1 = 0$ we clearly have $\tau_1 \in (\xi_0, \xi_1)$; hence we may set $\tau_{i_1} := \tau_1$. Suppose now that $\tau_{i_1}, \ldots, \tau_{i_s}$ have been chosen $(s < n+r)$. We define

$$\tau_{i_{s+1}} := \min\{\tau_i \colon \tau_i > \tau_{i_s},\ \tau_i \in (\xi_{s+1-n}, \xi_{s+1})\}.$$

To demonstrate that such a choice is always possible, suppose that (ξ_{s+1-n}, ξ_{s+1}) contains no $\tau_i > \tau_{i_s}$. By the definition of τ_{i_s}, (ξ_{s-n}, ξ_s) contains no $\tau_i < \tau_{i_s}$. Thus, (ξ_{s+1-n}, ξ_s) contains at most one point, τ_{i_s}. However, $U := S_{n,r}|_{[\xi_{s+1-n}, \xi_s]}$ is a WT-space of dimension $2n-2$ such that $\int_a^b hu = 0$ for all $u \in U$. Since $n \ge 2$, h must have at least 2 sign changes in (ξ_{s+1-n}, ξ_s), a contradiction. This completes the proof of Proposition 1. $\square$

The following proposition, from [**15**, p. 363], is fundamental to the proof of our main result.

PROPOSITION 2. *Let K be a wedge in $L_1[a,b]$, and let $g_0 \in K$ and $f \in L_1[a,b]$ be given. If* $\operatorname{meas}\{f = g_0\} = 0$ *then g_0 is a best L_1-approximation to f if and only if*

$$\int_a^b \operatorname{sgn}(f - g_0)g \leq 0 \quad \textit{for all } g \in K, \tag{1.3}$$

and

$$\int_a^b \operatorname{sgn}(f - g_0)g_0 = 0. \tag{1.4}$$

In order to prove Theorem 1 below we will also make use of the following proposition.

PROPOSITION 3. *For $h \in L_\infty[a,b]$ define*

$$P(t) := \int_a^b h(x)\frac{(x-t)_+^{n-1}}{(n-1)!}\,dx. \tag{1.5}$$

Then

$$P^{(i)}(a) = 0 \ (i = 0, \ldots, n-1) \Leftrightarrow \int_a^b hq = 0 \quad \textit{for all } q \in \Pi_{n-1}, \tag{1.6}$$

$$P \leq 0 \quad \textit{and} \quad P^{(i)}(a) = 0 \ (i = 0, \ldots, n-1) \Leftrightarrow \int_a^b hg \leq 0 \quad \textit{for all } g \in K_n. \tag{1.7}$$

PROOF. Note that (1.6) follows immediately from

$$P^{(i)}(t) = (-1)^i \int_a^b h(x)\frac{(x-t)_+^{n-1-i}}{(n-1-i)!}\,dx$$

for $i = 0, \ldots, n-1$.

Suppose that $P \leq 0$ and that $P^{(i)}(a) = 0$ $(i = 0, \ldots, n-1)$. Let $g \in K_n$ be given. If μ is the measure associated with g, then from (1.6) and the convergence properties of $\{g_k\}_{k=1}^\infty := \{g_{a+1/k,b-1/k}\}_{k=1}^\infty$ mentioned above, Fubini's Theorem yields

$$\begin{aligned} \int_a^b P(t)\,d\mu(t) &= \lim_{k\to\infty} \int_{a+1/k}^{b-1/k} P(t)\,d\mu(t) \\ &= \lim_{k\to\infty} \int_a^b h(x)g_k(x)\,dx \\ &= \int_a^b h(x)g(x)\,dx. \end{aligned} \tag{1.8}$$

Thus, if $P \leq 0$, then $\int_a^b hg \leq 0$. Conversely, if $\int_a^b hg \leq 0$ for all $g \in K_n$ then, necessarily, $\int_a^b hq = 0$ for all $q \in \Pi_{n-1}$, and hence $P^{(i)}(a) = 0$ $(i = 0, \ldots, n-1)$. From (1.8) it follows that $P \leq 0$. □

REMARK 1. If $|h| = 1$ a.e. and h is equivalent in $L_\infty[a,b]$ to a function with only a finite number of sign changes, then the function P defined in (1.5) is an example of a *perfect spline* of degree n with knots at these sign changes [**2**].

2. Main results. We now present the main results of this paper. We give necessary and sufficient conditions for an n-convex function g_0 to be a best L_1-approximation to a function $f \in C[a,b]$ when certain assumptions about $f - g_0$ are made.

THEOREM 1. *Let $f \in C[a,b]$ and $g_0 \in K_n$ be given, and assume that* $\operatorname{meas}\{f = g_0\} = 0$ *and that $f - g_0$ has a finite number of sign changes $\tau_1 < \cdots < \tau_N$ in (a,b). Let P be as in* (1.5) *with $h := \operatorname{sgn}(f - g_0)$. Then g_0 is a best L_1-approximation to f from K_n if and only if* (2.1)–(2.3) *are satisfied.*

(2.1) $P \leq 0$;

(2.2) $P^{(i)}(a) = 0$ $(i = 0, \ldots, n-1)$;

(2.3) $g_0 \in S_{n,r}(\xi_1, \ldots, \xi_r)$, *where $\xi_1, \ldots, \xi_r$ are the distinct zeros of P in (a,b). Furthermore, if g_0 satisfies* (2.1)–(2.3) *then it is the unique best L_1-approximation to f from K_n.*

PROOF. We show that (2.1)–(2.3) are equivalent to (1.3) and (1.4). Since, by Proposition 3, (2.1) and (2.2) are equivalent to (1.3), it suffices to show that (1.4) is equivalent to (2.3), given either (2.1) and (2.2) or (1.3).

(2.3)$\Rightarrow$(1.4). Note that $h = (-1)^n P^{(n)}$ a.e. and that $P^{(i)}(b) = 0$ $(i = 0, \ldots, n-1)$. Thus, integration by parts yields

$$\int_a^b h(x) \frac{(x-\xi_i)_+^{n-1}}{(n-1)!}\,dx = \int_a^b (-1)^n P^{(n)}(x) \frac{(x-\xi_i)_+^{n-1}}{(n-1)!}\,dx$$
$$= -\int_a^b P'(x)(x-\xi_i)_+^0\,dx = -(P(b) - P(\xi_i)) = 0$$

for all $\xi_1, \ldots, \xi_r$. Thus, from (2.2) and (1.6) it follows that

$$\int_a^b hs = 0 \quad \text{for all } s \in S_{n,r}(\xi_1, \ldots, \xi_r).$$

In particular, (1.4) holds.

(1.4)$\Rightarrow$(2.3). From (1.8) we obtain

$$\int_a^b P(t)\,d\mu_0(t) = 0,$$

where μ_0 is the measure associated with g_0. Since $(x-t)_+^{n-1}$ is n-convex for all $t \in [a,b]$, (1.3) implies $P(t) \leq 0$, and therefore from the nonnegativity of μ_0 we have

$$\operatorname{supp} \mu_0 \subset P^{-1}\{0\}. \tag{2.4}$$

Since $P^{(n)}$ has N sign changes, P has at most $N + n$ zeros in $[a,b]$, counting multiplicities (see [**2**]). Thus, from (2.4) it follows that g_0 is a spline of degree $n-1$ with simple knots in the distinct zeros of P in (a,b), i.e., (2.3) is valid.

Uniqueness. For $g \in K_n$, (1.4) implies

$$\begin{aligned}\|f-g_0\|_1 &= \int_a^b \operatorname{sgn}(f-g_0)(f-g_0)\\ &= \int_a^b \operatorname{sgn}(f-g_0)(f-g) + \int_a^b \operatorname{sgn}(f-g_0)g\\ &\le \|f-g\|_1 + \int_a^b \operatorname{sgn}(f-g_0)g.\end{aligned}$$

Thus, if $g_1 \in K_n$ is another best L_1-approximation to f, then

$$\int_a^b \operatorname{sgn}(f-g_0)g_1 = 0, \quad \text{and} \tag{2.5}$$

$$\int_a^b \operatorname{sgn}(f-g_0)(f-g_1) = \|f-g_1\|_1. \tag{2.6}$$

From (2.6) it follows that $(f-g_0)(f-g_1) \ge 0$ on (a,b); hence $g_1(\tau_i) = g_0(\tau_i) = f(\tau_i)$ $(i = 1,\dots,N)$. Using (2.5), and reasoning as before, we get $g_1 \in S_{n,r}(\xi_1,\dots,\xi_r)$. Since, as shown above, $\int_a^b hs = 0$ for all $s \in S_{n,r}$, Proposition 1 implies that $g_1 - g_0 \equiv 0$, proving uniqueness.

This completes the proof of Theorem 1. □

REMARK 2. Let f, g_0 and P be as in Theorem 1.

(a) Let n be even. Then g_0 is a best approximation to f on any subinterval $[\alpha,\beta]$ such that P has a zero of order n in α and in β. One can show that such zeros do not coincide with $\xi_1,\dots,\xi_r$.

(b) If $P < 0$ in (a,b), then g_0 is an element of Π_{n-1}.

(c) From the structure of P, it follows that $N-n$ is even, say $N-n = 2k \ge 2r$. Moreover, since $(-1)^n P^{(n)} = \operatorname{sgn}(f-g_0)$ a.e., we must have $(-1)^{N-i}(f-g_0) \le 0$ in (τ_i,τ_{i+1}) $(i = 0,\dots,N)$.

For the special case $n = 2$ it is possible to make a more precise statement about the best L_1-approximation.

COROLLARY 1. *Let the conditions of Theorem* 1 *prevail, and set* $k := (N-2)/2$, $\xi_0 := a$ *and* $\xi_{k+1} := b$. *Then the following are equivalent*:

(a) g_0 *is a best* L_1*-approximation to* f *from* K_2 *and the corresponding* P *has* k *distinct zeros in* (a,b);

(b) $g_0 \in S_{2,k}(\xi_1,\dots,\xi_k)$ *and* g_0 *is a best convex approximation to* f *on* $[\xi_i,\xi_{i+1}]$ $(i = 0,\dots,k)$;

(c) $g_0 \in S_{2,k}(\xi_1,\dots,\xi_k)$, g_0 *is a best linear polynomial approximation to* f *on* $[\xi_i,\xi_{i+1}]$ $(i = 0,\dots,k)$ *and* $f-g_0$ *has exactly two sign changes in* (ξ_i,ξ_{i+1}) *with the last sign negative* $(i = 0,\dots,k)$;

(d) $g_0 \in S_{2,k}(\xi_1,\dots,\xi_k)$ *and* $f-g_0$ *changes sign precisely at* $\tau_{i,1} = \frac{1}{4}(3\xi_i+\xi_{i+1})$ *and at* $\tau_{i,2} = \frac{1}{4}(\xi_i+3\xi_{i+1})$ $(i = 0,\dots,k)$, *with the last sign negative.*

PROOF. (a)⇒(b). By its definition, P can vanish at most once in each interval (τ_{2i},τ_{2i+1}) $(i = 1,\dots,k)$, and not at all in the rest of (a,b). Thus, if P has k distinct zeros in (a,b), then the sign changes $\tau_1 < \cdots < \tau_{2k+2}$ of $f-g_0$ and the knots $\xi_1,\dots,\xi_k$ of g_0 must satisfy $\xi_i < \tau_{2i+1} < \tau_{2i+2} < \xi_{i+1}$ $(i = 1,\dots,k)$.

Since each of these zeros is a double zero, Theorem 1 implies that g_0 is a best approximation to f on $[\xi_i, \xi_{i+1}]$ $(i = 0, \ldots, k)$, and g_0 is a linear polynomial on each of these intervals since $P < 0$ in (ξ_i, ξ_{i+1}).

(b)$\Rightarrow$(c). Clearly, if g_0 is a best convex approximation to f on $[\xi_i, \xi_{i+1}]$, then it is also a best approximation from Π_1 there. Moreover, if g_0 is best on each subinterval then it is also best on all of $[a, b]$. The auxiliary function P for $[a, b]$ is pieced together from the P_i's corresponding to each $[\xi_i, \xi_{i+1}]$, which must contain at least two of the τ_i's. Since $N = 2k + 2$, there must be exactly two τ_i's in each (ξ_i, ξ_{i+1}), i.e., $f - g_0$ has two sign changes therein. That the last sign $f - g_0$ in (ξ_i, ξ_{i+1}) is negative follows from Remark 2c.

(c)$\Rightarrow$(d). We note that g_0 is a best approximation to f from Π_1 on $[\xi_i, \xi_{i+1}]$ if and only if

$$\int_{\xi_i}^{\xi_{i+1}} \operatorname{sgn}(f - g_0)q = 0 \quad \text{for all } q \in \Pi_1 \tag{2.7}$$

[**15**]. The two sign changes of $f - g_0$ may thus be computed directly, and are given by $\tau_{i,1}$ and $\tau_{i,2}$.

(d)$\Rightarrow$(a). Since g_0 is in $S_{2,k}(\xi_1, \ldots, \xi_k)$, it follows from (2.7) that (1.4) holds. For $0 \le i \le k$ and any convex function g, let $p \in \Pi_1$ be defined by $p(\tau_{i,1}) = g(\tau_{i,1})$ and $p(\tau_{i,2}) = g(\tau_{i,2})$. Due to the convexity of g and the assumption on $\operatorname{sgn}(f - g_0)$ we then have

$$\int_{\xi_i}^{\xi_{i+1}} \operatorname{sgn}(f - g_0)g = \int_{\xi_i}^{\xi_{i+1}} \operatorname{sgn}(f - g_0)(g - p) \le 0.$$

Thus, (1.3) is valid and g_0 is a best convex approximation to f (on each subinterval and hence on $[a, b]$). As constructed above, the function P vanishes at each ξ_i and thus has k distinct zeros in (a, b). $\square$

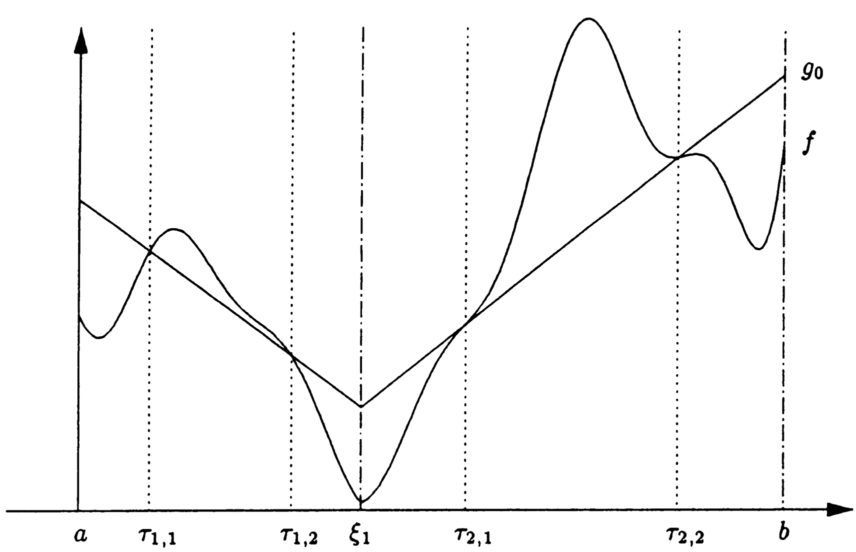

FIGURE 1. Best convex approximation to a continuous function $(n = 2, r = 1)$

EXAMPLE 1. The graph demonstrates Corollary 1. Here, g_0 is a linear spline with one knot, ξ_1. The points $\tau_{i,1}$ and $\tau_{i,2}$ for $i = 1, 2$ are defined as in Corollary 1(d) and $f - g_0$ changes sign precisely at these points with the last sign being negative. It follows that g_0 is the best convex approximation to f in the norm of $L_1[a, b]$.

We close with the following general remarks. The reason that g_0 is a spline of degree $n-1$ in Theorem 1 is, ultimately, because the functions $(\cdot - t)_+^{n-1}$ are the extreme rays, modulo Π_{n-1}, of K_n [**9**]. In the general case (without the restrictions on $f - g_0$), it may be shown that g_0 is a spline of degree $n-1$ with a specified number of knots on connected components of $\{f \neq g_0\}$. These splines are extremal solutions to a certain interpolation problem and are described in [**12**] (for $n = 2$ see [**8**]). This observation is valid for all $1 < p < \infty$ as well. Further, a theorem analogous to Theorem 1 holds in $L_p[a, b]$ $(1 < p < \infty)$, provided $h := \operatorname{sgn}(f - g_0)|f - g_0|^{p-1}$ has only a finite number of zeros. Uniqueness results from the uniform convexity of the norm. For $p = \infty$ see [**1**].

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ADDED IN PROOF. Proposition 2 was also proved independently by F. Deutsch in his dissertation, *Some applications of functional analysis to approximation theory*, Brown University, 1965.

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