A COMMUTATOR ESTIMATE
FOR PSEUDO-DIFFERENTIAL OPERATORS
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ABSTRACT. For the commutator $B \cdot A - \text{Op}(ba)$ of two pseudo-differential operators $A$ and $B$ an estimate on weighted Sobolev spaces is proved under minimal regularity assumptions on the symbols $a$ and $b$.

1. Let $a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ two symbols. We impose the following conditions on $a$ and $b$. Let $N$ be a natural number and suppose that for all multi-indices $\alpha$ such that $|\alpha| \leq N$ there holds

\begin{align*}
(1) & \quad |D^\alpha b(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \\
(2) & \quad |D_x D^\alpha b(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-|\alpha|}
\end{align*}

if $i = 1, \ldots, n$. Suppose further that for all $\alpha$ such that $|\alpha| \leq n$ there holds

\begin{align*}
(3) & \quad |D^\alpha a(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \\
(4) & \quad |D_x D^\alpha a(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \\
(5) & \quad |D_x D^\alpha (a(x + h, \xi) - a(x, \xi))| \leq C\omega(|h|, |\xi|)(1 + |\xi|)^{-|\alpha|}.
\end{align*}

We suppose that for each $t > 0$ the function $\omega(\cdot, t)$ is increasing and concave and that the functions $\omega(t, \cdot)$ and $\Omega$ are almost increasing in the following sense. There exists a positive constant $C$ independent of $t$ such that

\begin{align*}
(6) & \quad \omega(t, r) \leq C \omega(t, s)
\end{align*}

whenever $0.5 \leq r \leq 2s$ and similarly for $\Omega$.

2. Let $w$ be a positive, locally integrable function. We say that $w \in A_p$, i.e. $w$ satisfies Muckenhoupt’s $A_p$-condition for some $1 < p < \infty$, iff

\begin{align*}
(7) & \quad \sup_{Q} \frac{1}{|Q|} \int_Q w \left( \frac{1}{|Q|} \int_Q w^{-1/(p-1)} \right)^{p-1} < \infty
\end{align*}

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Denote by $L^p(w)$ the weighted $L^p$-space and let $J^s$ be the Bessel potential of order $s \in \mathbb{R}$. The weighted Sobolev space $H^{s,p}(w)$ is defined to be the space of all tempered distributions $f$ such that

\begin{align*}
(8) & \quad \|f\|_{H^{s,p}(w)} := \|J^{-s}f\|_{L^p(w)} < \infty
\end{align*}

(compare Miler [7]).

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3. Our main objective is to prove the following result.

**THEOREM 1.** Let $N = n + \lfloor n/2 \rfloor + 2$ and suppose that the symbols $a$ and $b$ satisfy (1), (2) and (3), (4), (5) respectively. Let $\omega$ and $\Omega$ satisfy (6) and be such that $\{2^{-j}\Omega(2^j)\}, \{\omega(2^{-j}, 2^j)\} \in l^2(N)$. Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < s \leq 1$. Then the commutator

$$B \cdot A - \text{Op}(ba): H^{s-1, p}(w) \to H^{s, p}(w)$$

is bounded. \(\square\)

This theorem extends earlier results by Kumano Go and Nagase [3] and Bourdau [1]; see also Marschall [4].

4. Before we prove the theorem let us provide a result needed in the proof. Let $c: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a symbol and suppose that for all multi-indices $\alpha$ such that $|\alpha| \leq n$ there holds

$$|D_\xi^\alpha c(x, \xi)| \leq C (1 + |\xi|)^{-|\alpha|},$$

$$|D_\xi^\alpha (c(x + h, \xi) - c(x, \xi))| \leq C \tilde{\omega}(|h|, |\xi|)(1 + |\xi|)^{-|\alpha|}.$$ 

**PROPOSITION 2.** Let $1 < p < \infty$ and $w \in A_p$.

(i) If the symbol $c$ satisfies (9) and (10) with a function $\tilde{\omega}$ such that (6) and $\{\tilde{\omega}(2^{-j}, 2^j)\} \in l^2(N)$ hold, then the operator $C: L^p(w) \to L^p(w)$ is bounded.

(ii) Suppose that (9) holds and that for each $\xi \in \mathbb{R}^n$ the function $c(\cdot, \xi)$ has its spectrum contained in the ball $\{\eta: |\eta| \leq 0.1 \times (1 + |\xi|^2)^{1/2}\}$. Then for every real number $s$ the operator $C: H^{s, p}(w) \to H^{s, p}(w)$ is bounded. \(\square\)

For a proof of the proposition see Marschall [5] and also Coifman and Meyer [2].

5. We are now in the position for the

**PROOF OF THEOREM.**

**Step (i).** Let $K$ be a function belonging to the Schwartz space $S(\mathbb{R}^n)$ such that the spectrum of $K$ is contained in the ball $B(0, 0.05)$ and that the Fourier transform of $K$ is equal to one in a neighborhood of the origin. Define

$$K_\xi(x) := (1 + |\xi|^2)^{n/2} K((1 + |\xi|^2)^{1/2} x)$$

and decompose the symbols $a$ and $b$ as follows. Let

$$a_1(x, \xi) := \int K_\xi(y) a(x - y, \xi) \, dy,$$

$$b_1(x, \xi) := \int K_\xi(y) b(x - y, \xi) \, dy$$

and $a_2 := a - a_1$ and $b_2 := b - b_1$. Note that by the conditions on the Fourier transform of $K$ one has

$$\int K_\xi(y) \, dy = 1, \quad \int y_i K_\xi(y) \, dy = 0, \quad i = 1, \ldots, n.$$

Hence, it follows that

$$|D_\xi b_2(x, \xi)| \leq C \ast \Omega(|\xi|)(1 + |\xi|)^{-1-|\alpha|},$$

$$|D_x, D_\xi^2 b_2(x, \xi)| \leq C \ast \Omega(|\xi|)(1 + |\xi|)^{-|\alpha|},$$

$$|D_\xi^2 a_2(x, \xi)| \leq C \omega((1 + |\xi|)^{-1}, |\xi|)(1 + |\xi|)^{-|\alpha|},$$

$$|D_x, D_\xi^2 a_2(x, \xi)| \leq C \omega((1 + |\xi|)^{-1}, |\xi|)(1 + |\xi|)^{-|\alpha|}.$$
For example, in order to prove (13) observe that
\[
a_2(x, \xi) = \int K_\xi(y) \left( a(x, \xi) - a(x - y, \xi) - \sum_{i=1}^{n} y_i \frac{\partial a}{\partial x_i}(x, \xi) \right) dy
\]
and hence, by the mean value theorem
\[
|a_2(x, \xi)| \leq C \int |y| |K_\xi(y)| \omega(|y|, |\xi|) dy
\leq C(1 + |\xi|)^{-1} \int |y| |K(y)| \omega((1 + |\xi|)^{-1}|y|, |\xi|) dy.
\]
Now, \(\omega(\cdot, |\xi|)\) being increasing and concave, one has
\[
\omega((1 + |\xi|)^{-1}|y|, |\xi|) \leq (1 + |y|)\omega((1 + |\xi|)^{-1}, |\xi|)
\]
(compare Coifman and Meyer [2]) and (13) follows.

**Step (ii).** By complex interpolation it suffices to prove the theorem in the endpoint cases \(s = 0\) and \(s = 1\). Consider first the term \(C_i := B_i \cdot A_1 - \text{Op}(b_i a_1), i = 1, 2\). It has the symbol
\[
c_i(x, \xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^{\infty} \int_0^1 \int \frac{\partial b_i}{\partial \xi_j}(x, \xi + t\eta)(D_{x,j} a_1)^\wedge(\eta, \xi) d\eta dt
\]
where \((D_{x,j} a_1)^\wedge(\cdot, \xi)\) is meant to be the Fourier transform of the function \(D_{x,j} a_1(\cdot, \xi)\). Then, using a method by Meyer [6], it follows that for all multi-indices \(\alpha\) such that \(|\alpha| \leq n\) one has
\[
|D_{\xi}^\alpha c_1(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|},
|D_{\xi}^\alpha c_2(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-2-|\alpha|},
|D_{x,j} D_{\xi}^\alpha c_2(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-1-|\alpha|}.
\]
But then the proposition and the condition \(\{2^{-j}\Omega(2^j)\} \in l^2(\mathbb{N})\) yield the boundedness of
\[
C_i: H^{s-1,p}(w) \rightarrow H^{s,p}(w).
\]

**Step (iii).** Observe that part (i) of the proposition and the condition \(\{2^{-j}\Omega(2^j)\} \in l^2(\mathbb{N})\) imply
\[
B: L^p(w) \rightarrow L^p(w),
B: H^{1,p}(w) \rightarrow H^{1,p}(w).
\]
Further, the proposition and (13), (14) yield
\[
A_2: H^{-1,p}(w) \rightarrow L^p(w),
A_2: L^p(w) \rightarrow H^{1,p}(w)
\]
and consequently we get the boundedness of the term \(B \cdot A_2\). Since the remaining term \(\text{Op}(b a_2)\) is treated similarly, the theorem is proved completely. \(\square\)

6. Let us remark that one can introduce symbols \(b\) with \(N\) not an integer. Then using complex interpolation of symbols one can see that the theorem holds when \(N = 3n/2 + 1\). For such techniques we refer to Marschall [4].
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