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A COMMUTATOR ESTIMATE FOR PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. For the commutator $B \cdot A - \mathrm{Op}(ba)$ of two pseudo-differential operators A and B an estimate on weighted Sobolev spaces is proved under minimal regularity assumptions on the symbols a and b.

1. Let a, b: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ two symbols. We impose the following conditions on a and b. Let N be a natural number and suppose that for all multi-indices α such that $|\alpha| \leq N$ there holds

$$|D_\xi^\alpha b(x,\xi)| \leq C(1+|\xi|)^{-|\alpha|}, \tag{1}$$

$$|D_{x_i} D_\xi^\alpha b(x,\xi)| \leq C * \Omega(|\xi|)(1+|\xi|)^{-|\alpha|} \tag{2}$$

if $i = 1, \ldots, n$. Suppose further that for all α such that $|\alpha| \leq n$ there holds

$$|D_\xi^\alpha a(x,\xi)| \leq C(1+|\xi|)^{-|\alpha|}, \tag{3}$$

$$|D_{x_i} D_\xi^\alpha a(x,\xi)| \leq C(1+|\xi|)^{-|\alpha|}, \tag{4}$$

$$|D_{x_i} D_\xi^\alpha (a(x+h,\xi) - a(x,\xi))| \leq C\omega(|h|, |\xi|)(1+|\xi|)^{-|\alpha|}. \tag{5}$$

We suppose that for each $t > 0$ the function $\omega(\cdot, t)$ is increasing and concave and that the functions $\omega(t, \cdot)$ and Ω are almost increasing in the following sense. There exists a positive constant C independent of t such that

$$\omega(t,\tau) \leq C\omega(t,s) \tag{6}$$

whenever $0.5 * \tau \leq s \leq 2 * \tau$ and similarly for Ω.

2. Let w be a positive, locally integrable function. We say that $w \in A_p$, i.e. w satisfies Muckenhoupt's A_p-condition for some $1 < p < \infty$, iff

$$\sup \frac{1}{|Q|} \int_Q w\,dx \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} dx \right)^{p-1} < \infty \tag{7}$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$. Denote by $L^p(w)$ the weighted L^p-space and let J^s be the Bessel potential of order $s \in \mathbf{R}$. The weighted Sobolev space $H^{s,p}(w)$ is defined to be the space of all tempered distributions f such that

$$\|f\|_{H^{s,p}(w)} := \|J^{-s} f\|_{L^p(w)} < \infty \tag{8}$$

(compare Miler [**7**]).

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3. Our main objective is to prove the following result.

THEOREM 1. *Let $N = n + [n/2] + 2$ and suppose that the symbols a and b satisfy* (1), (2) *and* (3), (4), (5) *respectively. Let ω and Ω satisfy* (6) *and be such that $\{2^{-j}\Omega(2^j)\}$, $\{\omega(2^{-j}, 2^j)\} \in l^2(\mathbf{N})$. Suppose that $1 < p < \infty$, $w \in A_p$ and $0 \leq s \leq 1$. Then the commutator*

$$B \cdot A - \mathrm{Op}(ba) \colon H^{s-1,p}(w) \to H^{s,p}(w)$$

is bounded. □

This theorem extends earlier results by Kumano Go and Nagase [**3**] and Bourdaud [**1**]; see also Marschall [**4**].

4. Before we prove the theorem let us provide a result needed in the proof. Let $c \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ be a symbol and suppose that for all multi-indices α such that $|\alpha| \leq n$ there holds

$$|D_\xi^\alpha c(x,\xi)| \leq C(1+|\xi|)^{-|\alpha|}, \tag{9}$$

$$|D_\xi^\alpha (c(x+h,\xi) - c(x,\xi))| \leq C\tilde{\omega}(|h|,|\xi|)(1+|\xi|)^{-|\alpha|}. \tag{10}$$

PROPOSITION 2. *Let $1 < p < \infty$ and $w \in A_p$.*

(i) *If the symbol c satisfies* (9) *and* (10) *with a function $\tilde{\omega}$ such that* (6) *and $\{\tilde{\omega}(2^{-j}, 2^j)\} \in l^2(\mathbf{N})$ hold, then the operator $C \colon L^p(w) \to L^p(w)$ is bounded.*

(ii) *Suppose that* (9) *holds and that for each $\xi \in \mathbf{R}^n$ the function $c(\cdot,\xi)$ has its spectrum contained in the ball $\{\eta \colon |\eta| \leq 0.1 * (1+|\xi|^2)^{1/2}\}$. Then for every real number s the operator $C \colon H^{s,p}(w) \to H^{s,p}(w)$ is bounded.* □

For a proof of the proposition see Marschall [**5**] and also Coifman and Meyer [**2**].

5. We are now in the position for the

PROOF OF THE THEOREM.

Step (i). Let K be a function belonging to the Schwartz space $S(\mathbf{R}^n)$ such that the spectrum of K is contained in the ball $B(0, 0.05)$ and that the Fourier transform of K is equal to one in a neighborhood of the origin. Define

$$K_\xi(x) := (1+|\xi|^2)^{n/2} K((1+|\xi|^2)^{1/2} x)$$

and decompose the symbols a and b as follows. Let

$$a_1(x,\xi) := \int K_\xi(y) a(x-y,\xi)\,dy,$$

$$b_1(x,\xi) := \int K_\xi(y) b(x-y,\xi)\,dy$$

and $a_2 := a - a_1$ and $b_2 := b - b_1$. Note that by the conditions on the Fourier transform of K one has

$$\int K_\xi(y)\,dy = 1, \qquad \int y_i K_\xi(y)\,dy = 0, \qquad i = 1,\ldots,n.$$

Hence, it follows that

$$|D_\xi^\alpha b_2(x,\xi)| \leq C * \Omega(|\xi|)(1+|\xi|)^{-1-|\alpha|}, \tag{11}$$

$$|D_{x_i} D_\xi^\alpha b_2(x,\xi)| \leq C * \Omega(|\xi|)(1+|\xi|)^{-|\alpha|}, \tag{12}$$

$$|D_\xi^\alpha a_2(x,\xi)| \leq C\omega((1+|\xi|)^{-1},|\xi|)(1+|\xi|)^{-1-|\alpha|}, \tag{13}$$

$$|D_{x_i} D_\xi^\alpha a_2(x,\xi)| \leq C\omega((1+|\xi|)^{-1},|\xi|)(1+|\xi|)^{-|\alpha|}. \tag{14}$$

For example, in order to prove (13) observe that

$$a_2(x,\xi) = \int K_\xi(y) \left\langle a(x,\xi) - a(x-y,\xi) - \sum_{i=1}^n y_i \frac{\partial a}{\partial x_i}(x,\xi) \right\rangle dy$$

and hence, by the mean value theorem

$$\begin{aligned} |a_2(x,\xi)| &\le C \int |y|\,|K_\xi(y)|\omega(|y|,|\xi|)\,dy \\ &\le C(1+|\xi|)^{-1} \int |y|\,|K(y)|\omega((1+|\xi|)^{-1}|y|,|\xi|)\,dy. \end{aligned}$$

Now, $\omega(\cdot,|\xi|)$ being increasing and concave, one has

$$\omega((1+|\xi|)^{-1}|y|,|\xi|) \le (1+|y|)\omega((1+|\xi|)^{-1},|\xi|)$$

(compare Coifman and Meyer [**2**]) and (13) follows.

Step (ii). By complex interpolation it suffices to prove the theorem in the endpoint cases $s=0$ and $s=1$. Consider first the term $C_i := B_i \cdot A_1 - \mathrm{Op}(b_i a_1)$, $i=1,2$. It has the symbol

$$c_i(x,\xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^n \int_0^1 \int \frac{\partial b_i}{\partial \xi_j}(x,\xi+t\eta)(D_{x_j}a_1)^\wedge(\eta,\xi)\,d\eta\,dt$$

where $(D_{x_j}a_1)^\wedge(\cdot,\xi)$ is meant to be the Fourier transform of the function $D_{x_j}a_1(\cdot,\xi)$. Then, using a method by Meyer [**6**], it follows that for all multi-indices α such that $|\alpha| \le n$ one has

$$\begin{aligned} &|D_\xi^\alpha c_1(x,\xi)| \le C(1+|\xi|)^{-|\alpha|}, \\ &|D_\xi^\alpha c_2(x,\xi)| \le C * \Omega(|\xi|)(1+|\xi|)^{-2-|\alpha|}, \\ &|D_{x_j} D_\xi^\alpha c_2(x,\xi)| \le C * \Omega(|\xi|)(1+|\xi|)^{-1-|\alpha|}. \end{aligned}$$

But then the proposition and the condition $\{2^{-j}\Omega(2^j)\} \in l^2(\mathbf{N})$ yield the boundedness of

$$C_i\colon H^{s-1,p}(w) \to H^{s,p}(w). \tag{15}$$

Step (iii). Observe that part (i) of the proposition and the condition $\{2^{-j}\Omega(2^j)\} \in l^2(\mathbf{N})$ imply

$$B\colon L^p(w) \to L^p(w), \tag{16}$$

$$B\colon H^{1,p}(w) \to H^{1,p}(w). \tag{17}$$

Further, the proposition and (13), (14) yield

$$A_2\colon H^{-1,p}(w) \to L^p(w), \tag{18}$$

$$A_2\colon L^p(w) \to H^{1,p}(w) \tag{19}$$

and consequently we get the boundedness of the term $B \cdot A_2$. Since the remaining term $\mathrm{Op}(ba_2)$ is treated similarly, the theorem is proved completely. □

6. Let us remark that one can introduce symbols b with N not an integer. Then using complex interpolation of symbols one can see that the theorem holds when $N = 3n/2+1$. For such techniques we refer to Marschall [**4**].

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