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NONSTANDARD METHODS FOR FAMILIES OF τ-SMOOTH PROBABILITY MEASURES

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ABSTRACT. For families of τ-smooth probability measures we give nonstandard characterizations of uniform τ-smoothness, uniform tightness, uniform pretightness, and relative compactness in the weak topology. We apply these characterizations to obtain two important theorems of probability theory: A theorem of Topsøe and a theorem of Prohorov.

1. Introduction and notation. Nonstandard analysis has proved to be extremely useful for measure and probability theory in recent years; see e.g. the excellent survey of Cutland [**3**]. On the one hand nonstandard methods allow very beautiful and illuminating proofs of classical results in this area. On the other hand measure theoretical concepts in the standard world have very intuitive correspondences in the nonstandard world.

Let $\mathscr{P}$ be a family of probability measures (p-measures) on the Borel-field $\mathscr{B}$ of a topological space X. In this paper we consider some important concepts of measure and probability theory for such families $\mathscr{P}$: uniform τ-smoothness, uniform tightness, uniform pretightness and relative compactness in the weak topology. We show that to each of these standard properties for $\mathscr{P}$ there corresponds a "nonstandard" subset of ${}^*X$ such that $\mathscr{P}$ has this standard property if and only if the corresponding subset of ${}^*X$ has full outer measure for each $Q \in {}^*\mathscr{P}$. Therefore relations between properties of $\mathscr{P}$ become "inclusions" between the corresponding subsets of ${}^*X$. This correspondence is used to obtain Topsøe's results that $\mathscr{P}$ is uniformly τ-smooth if and only if $\mathscr{P}$ is relatively compact in the family of all τ-smooth measures, as well as Prohorov's result that in a complete metric space relative compactness of $\mathscr{P}$ implies uniform tightness.

Let $\mathscr{W}$ be the system of all p-measures on the Borel-field $\mathscr{B}$ of a Hausdorff-space $(X,\mathscr{T})$. $P \in \mathscr{W}$ is called τ-smooth iff for each upwards directed system $\mathscr{S} \subset \mathscr{T}$ of open sets: $P(\bigcup_{S\in\mathscr{S}} S) = \sup_{S\in\mathscr{S}} P(S)$. Denote by $\mathscr{W}_\tau$ the system of all τ-smooth $P \in \mathscr{W}$.

A family $\mathscr{P} \subset \mathscr{W}$ is called

(i) *uniformly τ-smooth in* X iff $\mathscr{T} \supset \mathscr{S} \uparrow X$ implies $\sup_{S\in\mathscr{S}} \inf_{P\in\mathscr{P}} P(S) = 1$;

(ii) *uniformly tight* iff $\sup_{K\in\mathscr{K}} \inf_{P\in\mathscr{P}} P(K) = 1$, where $\mathscr{K}$ is the system of compact sets of X;

(iii) *uniformly pretight* iff $\sup_{C\in\mathscr{C}} \inf_{P\in\mathscr{P}} P(C) = 1$, where $\mathscr{C}$ is the system of all closed and totally bounded sets of a uniform space X.

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Let X be a completely regular Hausdorff space and $C_b(X)$ the family of all bounded and continuous functions. The weak topology for $\mathscr{W}$ is the smallest topology rendering the functions

$$P \in \mathscr{W} \to P[f] := \int f(x)P(dx) \in \mathbf{R}$$

continuous for all $f \in C_b(X)$. If $P \in \mathscr{W}$ and $Q \in {}^*\mathscr{W}$ we write $P \approx Q$ if P is infinitesimal close to Q with respect to the weak topology. It is well known that

$$(1.1) \qquad P \approx Q \Leftrightarrow P[f] \approx Q[{}^*f] \quad \text{for all } f \in C_b(X).$$

In this paper we consider a polysaturated nonstandard model including the real numbers $\mathbf{R}$ and the set X. Let

$$\mathrm{ns}^*X = \bigcup_{x \in X} \bigcap \{{}^*T \colon x \in T \in \mathscr{T}\}$$

be the system of all near-standard points of ${}^*X$ and let

$$\mathrm{cpt}^*X = \bigcup \{{}^*K \colon K \subset X,\ K \text{ compact}\}$$

be the system of all compact points of ${}^*X$. Let X be a uniform space with uniformity $\mathscr{U}$. Then

$$\mathrm{pns}^*X = \{x \in^* X \colon y \approx x \Rightarrow (x,y) \in \bigcap \{{}^*V : V \in \mathscr{U}\}\}$$

is the system of all pre-near-standard points of ${}^*X$ and

$$\mathrm{pcpt}^*X = \bigcup \{{}^*C \colon C \subset X \text{ closed and totally bounded}\}$$

is the system of all precompact points of ${}^*X$.

The following inclusions are well known

$$(1.2) \qquad \mathrm{pns}^*X \supset \mathrm{pcpt}^*X \supset \mathrm{cpt}^*X \subset \mathrm{ns}^*X \subset \mathrm{pns}^*X$$

(see [**7**, Chapters 8.3 and 8.4]). Furthermore

$$(1.3) \qquad X \text{ complete } \Rightarrow \mathrm{ns}^*X = \mathrm{pns}^*X \text{ and } \mathrm{cpt}^*X = \mathrm{pcpt}^*X$$

(see [**7**, 8.4.28]).
By $\mathrm{st} = \mathrm{st}_X$ we denote the standard part map from ns^*X into X. Let Q be a finite internal content on ${}^*\mathscr{B}$. Put for $A \subset^* X$:

$$\underline{Q}(A) = \sup\{\mathrm{st}_{\mathbf{R}} Q(B) \colon A \supset B \in^* \mathscr{B}\},$$
$$\overline{Q}(A) = \inf\{\mathrm{st}_{\mathbf{R}} Q(B) \colon A \subset B \in^* \mathscr{B}\}.$$

Then $L(Q,{}^*\mathscr{B}) = \{A \subset {}^*X \colon \underline{Q}(A) = \overline{Q}(A)\}$ is a σ-field over ${}^*X$ and $\overline{Q}|L(Q,{}^*\mathscr{B})$ is the Loeb-measure of Q; a concept which is of considerable importance for nonstandard measure theory (see [**5**]). Put

$$\overline{Q}_{\mathrm{st}}(A) = \overline{Q}(\mathrm{st}^{-1}A) \quad \text{for } A \subset X.$$

2. The results. We write $\overline{{}^*\mathscr{P}}(A) = 1$ iff $\overline{Q}(A) = 1$ for all $Q \in {}^*\mathscr{P}$.

2.1 THEOREM. *Let X be a topological space which is assumed to be Hausdorff for* (i) + (ii), *to be metric for* (iii) *and to be a completely regular Hausdorff space for* (iv). *Let $\mathscr{P}$ be a family of p-measures on the Borel-field of X. Then*

(i) *$\mathscr{P}$ uniformly τ-smooth in $X \Leftrightarrow \overline{{}^*\mathscr{P}}(\mathrm{ns}^*X) = 1$;*

(ii) *$\mathscr{P}$ uniformly tight $\Leftrightarrow \overline{{}^*\mathscr{P}}(\mathrm{cpt}^*X) = 1$;*

(iii) *$\mathscr{P}$ uniformly pretight $\Leftrightarrow \overline{{}^*\mathscr{P}}(\mathrm{pcpt}^*X) = 1 \Leftrightarrow \overline{{}^*\mathscr{P}}(\mathrm{pns}^*X) = 1$;*

(iv) *$\mathscr{P} \subset \mathscr{W}_\tau$: $\mathscr{P}$ relatively compact in $\mathscr{W}_\tau \Leftrightarrow \overline{{}^*\mathscr{P}}(\mathrm{ns}^*X) = 1$.*

PROOF. (i) $\Rightarrow$: Let $Q \in {}^*\mathscr{P}$ and $\mathrm{ns}^*X \subset B \in {}^*\mathscr{B}$; we have to show $Q(B) \approx 1$. For each $x \in X$ we have $\bigcap\{{}^*T\colon x \in T,\ T \text{ open}\} \subset \mathrm{ns}^*X \subset B$. Hence by saturation there exists an open set T_x with $x \in T_x$ and ${}^*T_x \subset B$. Let $\varepsilon \in \mathbf{R}_+$ be given. As $\{\bigcup_{x\in E} T_x\colon E \subset X \text{ finite}\} \uparrow X$ and $\mathscr{P}$ is uniformly τ-smooth in X, there exists a finite set $E \subset X$ such that $P(\bigcup_{x\in E} T_x) \geq 1-\varepsilon$ for all $P \in \mathscr{P}$. Hence $Q(B) \geq Q(\bigcup_{x\in E} {}^*T_x) \geq 1 - \varepsilon$ by transfer.

(i) $\Leftarrow$: Let $\mathscr{S}$ be a system of open sets with $\mathscr{S} \uparrow X$. Then $\mathrm{ns}^*X \subset \bigcup\{{}^*S\colon S \in \mathscr{S}\}$. Using our assumption we obtain by Theorem 1(ii) of [**4**]: $1 = \overline{Q}(\mathrm{ns}^*X) \leq \overline{Q}(\bigcup_{S\in\mathscr{S}} {}^*S) = \sup_{S\in\mathscr{S}} \overline{Q}({}^*S)$ for all $Q \in {}^*\mathscr{P}$. Hence ${}^*\mathscr{P} \subset \bigcup_{S\in\mathscr{S}}\{Q \in {}^*\mathscr{P}\colon Q({}^*S) \geq 1-\varepsilon\}$ for each $\varepsilon \in \mathbf{R}_+$. As our model is polysaturated and $\mathscr{S}\uparrow$, there exists $S \in \mathscr{S}$ such that $Q({}^*S) \geq 1 - \varepsilon$ for all $Q \in {}^*\mathscr{P}$. By transfer $P(S) \geq 1 - \varepsilon$ for all $P \in \mathscr{P}$.

(ii) Follows from Lemma 2.4 below applied to the system $\mathscr{K}$ of all compact sets.

(iii) The first equivalence follows from Lemma 2.4 applied to the system $\mathscr{C}$ of all totally bounded and closed sets. As $\mathrm{pcpt}^*X \subset \mathrm{pns}^*X$ it remains to show that $\overline{{}^*\mathscr{P}}(\mathrm{pns}^*X) = 1$ implies that $\mathscr{P}$ is uniformly pretight. To do this, it suffices to show that for each $\varepsilon \in \mathbf{R}_+$, $n \in \mathbf{N}$, there exists a closed set C which can be covered by finitely many spheres of radius $1/n$, fulfilling $\inf_{P\in\mathscr{P}} P(C) \geq 1-\varepsilon$. Let $K_\delta(x)$ be the closed sphere with center x and radius δ. As $\overline{Q}(\mathrm{pns}^*X) = 1$ for all $Q \in {}^*\mathscr{P}$ by assumption, and as $\mathrm{pns}^*X \subset \bigcup_{x\in X} {}^*K_{1/n}(x)$ we obtain

$$\overline{Q}\left(\bigcup_{x\in X} {}^*K_{1/n}(x)\right) = 1 \quad \text{for all } Q \in {}^*\mathscr{P}.$$

Hence by Theorem 1(ii) of [**4**] there exists for each $Q \in {}^*\mathscr{P}$ a finite set $E = E(Q) \subset X$ such that

$$Q\left(\bigcup_{x\in X} {}^*K_{1/n}(x)\right) \geq 1 - \varepsilon.$$

Therefore

$${}^*\mathscr{P} \subset \bigcup_{E \text{ finite}} \left\{Q \in {}^*\mathscr{P}\colon Q\left({}^*\left(\bigcup_{x\in E} K_{1/n}(x)\right)\right) \geq 1 - \varepsilon\right\}.$$

By saturation there exists a finite set $E_0 \subset X$ such that

$$Q\left({}^*\left(\bigcup_{x\in E_0} K_{1/n}(x)\right)\right) \geq 1 - \varepsilon \quad \text{for all } Q \in {}^*\mathscr{P}.$$

Put $C := \bigcup_{x\in E_0} K_{1/n}(x)$. Then $\inf_{P\in\mathscr{P}} P(C) \geq 1 - \varepsilon$ by transfer.

(iv) $\Rightarrow$: Let $Q \in {}^*\mathscr{P}$. Since $\mathscr{P}$ is relatively compact in the regular Hausdorff space $\mathscr{W}_\tau$, there exists according to 8.3.11 of [**7**] an element $P \in \mathscr{P}$ with $Q \approx P$

and hence $Q[^*f] \approx P[f]$ for all $f \in C_b(X)$ (see (1.1)). By Lemma 2.6 this implies $P|\mathscr{B} = \overline{Q}_{\mathrm{st}}|\mathscr{B}$ and hence $\overline{Q}(\mathrm{ns}^*X) = \overline{Q}_{\mathrm{st}}(X) = P(X) = 1$.

(iv) $\Leftarrow$: To prove that $\mathscr{P}$ is relatively compact in $\mathscr{W}_\tau$ it suffices to show that for each $Q \in {}^*\mathscr{P}$ ($\subset {}^*\mathscr{W}_\tau$) there exists $P \in \mathscr{W}_\tau$ with $Q \approx P$ (see 8.3.11 of [**7**]). By Theorem 4(ii) of [**4**], $P|\mathscr{B} = \overline{Q}_{\mathrm{st}}|\mathscr{B}$ is a τ-smooth measure with $P(X) = \overline{Q}(\mathrm{ns}^*X) = 1$ by assumption. Hence $P \in \mathscr{W}_\tau$. According to Lemma 2.6 we have $P \approx Q$.

Theorem 2.1 directly implies a result of Topsøe [**8**] and a result of Prohorov (see [**2**]).

2.2 COROLLARY. *Let X be a completely regular Hausdorff space and $\mathscr{P}$ be a family of τ-smooth p-measures. Then $\mathscr{P}$ is uniformly τ-smooth if and only if $\mathscr{P}$ is relatively compact in $\mathscr{W}_\tau$.*

PROOF. Direct consequence of Theorem 2.1(i)+(iv).

2.3 COROLLARY. *Let X be a complete metric space and $\mathscr{P}$ be a family of τ-smooth p-measures. Then $\mathscr{P}$ is uniformly tight if and only if $\mathscr{P}$ is relatively compact in $\mathscr{W}_\tau$.*

PROOF. As in a complete metric space $\mathrm{ns}^*X = \mathrm{pns}^*X$ (see (1.3)) and uniform tightness is the same as uniform pretightness, the assertion follows from Theorem 2.1(iii) and (iv).

Let $\mathscr{P}$ be a family of Radon-measures. With nonstandard techniques it was shown by Müller [**6**] for metric and by Anderson-Rashid [**1**] for rather general topological spaces that $\mathscr{P}$ is relatively compact in the family of all Radon-measures if it is uniformly tight. We proved in Corollary 2.3 the converse direction for complete metric space by nonstandard methods; observe that in a complete metric space the τ-smooth p-measures are exactly the Radon-p-measures.

Corollary 2.2 seems to be the first result for families of τ-smooth (not necessarily Radon-) measures which is proved by nonstandard techniques. The use of τ-smooth measures leads to additional difficulties, since for τ-smooth measures neither ns^*X nor the standard part map is Loeb-measurable in general. Both properties have been used heavily for the cited results on Radon-measures.

2.4 LEMMA. *Let $\mathscr{P}$ be a family of p-measures on a σ-field $\mathscr{A}$. Let $\mathscr{C} \subset \mathscr{A}$ be closed under finite unions. Then the following assertions are equivalent:*

(i) $\sup_{C\in\mathscr{C}} \inf_{P\in\mathscr{P}} P(C) = 1$.

(ii) $\overline{Q}(\bigcup\{^*C : C \in \mathscr{C}\}) = 1$ *for all* $Q \in {}^*\mathscr{P}$.

PROOF. By the transfer principle (i) implies (ii).

(ii)$\Rightarrow$(i). Assume indirectly that there exists $\varepsilon_0 \in \mathbf{R}_+$, such that $\mathscr{P}_C := \{P \in \mathscr{P} : P(C) \le 1 - \varepsilon_0\} \ne \varnothing$ for each $C \in \mathscr{C}$. Since $\mathscr{C}$ is closed under finite unions, the system $\{\mathscr{P}_C : C \in \mathscr{C}\}$ has the finite intersection property. By the enlargement property there exists $Q \in \bigcap\{^*\mathscr{P}_C : C \in \mathscr{C}\}$. Hence $Q(^*C) \le 1 - \varepsilon_0$ for all $C \in \mathscr{C}$. As $\{^*C : C \in \mathscr{C}\} \uparrow \bigcup\{^*C : C \in \mathscr{C}\}$ we obtain from Theorem 1(ii) of [**4**], that

$$\overline{Q}\left(\bigcup\{^*C : C \in \mathscr{C}\}\right) = \sup\{\overline{Q}(^*C) : C \in \mathscr{C}\} \le 1 - \varepsilon_0$$

contradicting (ii).

2.5 LEMMA. *Let X be a regular Hausdorff space with Borel-field $\mathscr{B}$ and let $Q \in {}^*\mathscr{W}$. Then*

$$\overline{Q}(\mathrm{ns}^*X) = 1 \Rightarrow Q[{}^*f] \approx \overline{Q}_{\mathrm{st}}[f] \quad \textit{for all } f \in C_b(X).$$

PROOF. Let $f \in C_b(X)$. According to Theorem 3.6 of [**3**] we have

$$Q[{}^*f] \approx \int \mathrm{st}_{\mathbf{R}}({}^*f(x))Q_L(dx). \tag{2.1}$$

$\overline{Q}(\mathrm{ns}^*X) = 1$ implies that $\overline{Q}|L(Q,{}^*\mathscr{B}) \cap \mathrm{ns}({}^*X)$ is a p-measure which is denoted by Q_0. As $f \in C_b(X)$ we obtain

$$\begin{aligned} \int \mathrm{st}_{\mathbf{R}}({}^*f(x))Q_L(dx) &= \int_{\mathrm{ns}^*X} \mathrm{st}_{\mathbf{R}}({}^*f(x))Q_0(dx) \\ &= \int_{\mathrm{ns}^*X} f(\mathrm{st}_X(x))Q_0(dx) = Q_0[f \circ \mathrm{st}]. \end{aligned} \tag{2.2}$$

Since $\mathrm{st}\colon \mathrm{ns}^*X \to X$ is $L(Q,{}^*\mathscr{B}) \cap \mathrm{ns}^*X$, $\mathscr{B}$-measurable by Corollary 3(iv) of [**4**], we obtain

$$Q_0[f \circ \mathrm{st}] = (Q_0)_{\mathrm{st}}[f] = \overline{Q}_{\mathrm{st}}[f]. \tag{2.3}$$

Now (2.1)–(2.3) imply the assertion.

2.6 LEMMA. *Let X be a completely regular Hausdorff space with Borel field $\mathscr{B}$. If $P \in \mathscr{W}_\tau$ and $Q \in {}^*\mathscr{W}_\tau$, then*

$$P[f] \approx Q[{}^*f] \quad \textit{for all } f \in C_b(X) \Leftrightarrow P|\mathscr{B} = \overline{Q}_{\mathrm{st}}|\mathscr{B}.$$

PROOF. $\Rightarrow$: At first we prove

$$\overline{Q}(\mathrm{ns}^*X) = 1. \tag{2.4}$$

Let $\mathrm{ns}^*X \subset B \in {}^*\mathscr{B}$. By saturation for each $x \in X$ there exists an open set T_x with $x \in T_x$ and ${}^*T_x \subset B$. Let $\varepsilon \in \mathbf{R}_+$ be given. As P is τ-smooth and $\{\bigcup_{x \in E} T_x\colon E \subset X \text{ finite}\} \uparrow X$, there exists a finite set $E \subset X$ such that

$$P(T) > 1 - \varepsilon \quad \text{where } T := \bigcup_{x \in E} T_x. \tag{2.5}$$

Since P is τ-smooth and X is completely regular we have

$$P(T) = \sup\{P[f]\colon 0 \le f \le 1_T,\ f \in C_b(X)\}. \tag{2.6}$$

By (2.5) and (2.6) there exists $f \in C_b(X)$ such that

$$f \le 1_T \quad \text{and} \quad P[f] \ge 1 - \varepsilon. \tag{2.7}$$

As $B \supset^* T$ we have by (2.7) and assumption

$$Q(B) \ge Q({}^*T) = Q[1_{{}^*T}] \underset{(2.7)}{\ge} Q[{}^*f] \approx P[f] \underset{(2.7)}{\ge} 1 - \varepsilon.$$

Hence $Q(B) \approx 1$, i.e. (2.4) is shown.
By (2.4) and Lemma (2.5) we obtain $Q[{}^*f] \approx \overline{Q}_{\mathrm{st}}[f]$ for all $f \in C_b(X)$. As by assumption $P[f] \approx Q[{}^*f]$ we have

$$P[f] = \overline{Q}_{\mathrm{st}}[f] \quad \text{for all } f \in C_b(X). \tag{2.8}$$

As P is τ-smooth, as $\overline{Q}_{\mathrm{st}}$ is τ-smooth by Theorem 4(ii) of [**4**] and as X is a completely regular Hausdorff space we obtain from (2.8) that $P|\mathscr{B} = \overline{Q}_{\mathrm{st}}|\mathscr{B}$.

$\Leftarrow$: As $P|\mathscr{B} = \overline{Q}_{\mathrm{st}}|\mathscr{B}$ we have $\overline{Q}(\mathrm{ns}^*X) = P(X) = 1$. Therefore Lemma 2.5 implies $Q[{}^*f] \approx \overline{Q}_{\mathrm{st}}[f] = P[f]$ for all $f \in C_b(X)$.

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