# AN UPPER BOUND FOR THE PROJECTION CONSTANT

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ABSTRACT. There is a positive function $\delta(n)$ of exponential order such that, for any normed space $E$ of dimension $n \geq 2$, the projection constant of $E$ satisfies $\lambda(E) \leq n^{1/2}[1 - \delta(n)]$.

The *projection constant* $\lambda(E)$ of a normed space $E$ is the smallest $\lambda > 0$ with the following property; whenever $X$ is a normed space with $E \subset X$ isometrically, there is a projection $u: X \to E$ with $\|u\| \leq \lambda$.

Kadec' and Snobar [2] first proved that $\lambda(E) \leq n^{1/2}$ for every $n$-dimensional space $E$. In [4] the strict inequality $\lambda(E) < n^{1/2}$ was verified in case $n \geq 2$, but the infinite dimensional proof used there produced no concrete estimates for $\lambda(E)$. Here we establish an upper estimate for $\lambda(E)$ and an associated lower estimate for $\pi_1(E)$ as well.

The *absolutely summing constant* of $E$, $\pi_1(E)$, is the smallest $c > 0$ such that

$$\sum_{i \leq m} \|x_i\| \leq c \max_{|\varepsilon_i|=1} \left\| \sum_{i \leq m} \varepsilon_i x_i \right\|$$

for all choices of $x_1, x_2, \ldots, x_m \in E$.

THEOREM 1. *Let $E$ be an $n$-dimensional space with $n \geq 2$. Then*
(a) $\pi_1(E) \geq n^{1/2} + (1/5)^{n+4}$, *and*
(b) $\lambda(E) \leq n^{1/2}[1 - n^{-2}(1/5)^{2n+11}]$.

Part (b) gives an exponential lower estimate for

$$\delta_n = \inf\{1 - n^{-1/2}\lambda(E): \dim E = n\},$$

but it is possible that there is actually a power type lower estimate, i.e., that $\delta_n \geq cn^{-p}$ for some positive $c$ and $p$. Complex examples constructed by H. König [3] show that $\delta_n \leq n^{-1/2}$ for infinitely many $n$.

Theorem 1 follows directly from an inequality relating the uniform and $L_1$-norms (Theorem 2) which will be proven first.

For the most part the notation and terminology is standard. Below $T$ always denotes a compact Hausdorff space and $C(T)$ the space of real-valued continuous functions on $T$ under the uniform norm $\|g\|_u$. For $\mu$ a regular Borel measure on $T$, $L_p(\mu)$ is the usual space of equivalence classes of functions for which $\|f\|_p = (\int |f|^p \, d\mu)^{1/p} < \infty$.

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THEOREM 2. *Let $E \subset C(T)$ be a subspace of finite dimension $n \geq 2$. If $\mu$ is a Borel probability on $T$ and $c$ a constant with $\|g\|_u \leq c\|g\|_1$ for all $g$ in $E$, then $c \geq n^{1/2} + (1/5)^{n+4}$.*

PROOF OF THEOREM 2. Write $E_1$ for $E$ considered as a subspace of $L_1(\mu)$. Applying Theorem 1 of [5] to $E_1$, there is a basis $f_1, f_2, \ldots, f_n \in E$ so that, if $f = (\sum_{i \leq n} f_i^2)^{1/2}$, then

$$(1) \qquad n \int f_i f_k f^{-1} \, d\mu = \delta_{ik}, \qquad 1 \leq i, \ k \leq n.$$

It follows that

$$(2) \qquad \int f \, d\mu = \sum_{i \leq n} \int f_i^2 f^{-1} \, d\mu = 1.$$

We also need the uniform bound

$$(3) \qquad f(s) \leq c, \quad \text{all } s \in T.$$

To see this, fix $s \in T$. The theorem's hypothesis gives

$$(4) \qquad f(s)^2 \leq \left\| \sum_{i \leq n} f_i(s) f_i \right\|_u \leq c \int \left| \sum_{i \leq n} f_i(s) f_i \right| \, d\mu.$$

The Cauchy-Schwartz inequality shows that $|\sum_{i \leq n} f_i(s) f_i| \leq f(s)f$ pointwise on $T$, hence

$$f(s)^2 \leq cf(s) \int f \, d\mu = cf(s).$$

Now let $d\nu = f \, d\mu$, so $\nu$ is a probability on $T$. Below $\| \ \|_p$ will denote the norm in $L_p(\nu)$. For $1 \leq i \leq n$ let $h_i = n^{1/2} f_i f^{-1}$. It follows from (1) that $h_1, \ldots, h_n$ is orthonormal in $L_2(\nu)$ and clearly $\sum_{i \leq n} h_i(s)^2 = n$ everywhere on $T$. We first claim that the kernel

$$h(s, t) = \sum_{i \leq n} h_i(s) h_i(t)$$

satisfies

$$(5) \qquad \iint |n^{-1/2}|h| - 1|^2 \, d\nu \, d\nu \leq 2[c - n^{1/2}].$$

To establish this first use (4); for $s \in T$

$$f(s) \leq c \int \left| \sum_{i \leq n} f_i(s) f(s)^{-1} f_i f^{-1} \right| f \, d\mu$$

$$= cn^{-1} \|h(s, \cdot)\|_1.$$

From orthogonality of the $h_i$'s

$$\|h(s, \cdot)\|_2 = \left( \sum_{i \leq n} h_i(s)^2 \right)^{1/2} = n^{1/2},$$

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and from the last two estimates

$$\|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 = n^{-1}\|h(s,\cdot)\|_2^2 - 2n^{-1/2}\|h(s,\cdot)\|_1 + 1$$
$$\leq 2[1 - n^{1/2}c^{-1}f(s)].$$

Integrating the last inequality, and taking (2) and (3) into account,

$$\int \|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \nu(ds) \leq c \int \|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \mu(ds)$$

$$\leq 2c[1 - n^{1/2}c^{-1} \int f(s)\mu(ds)] = 2[c - n^{1/2}].$$

Note that (5) implies $c \geq n^{1/2}$. Let $M$ be the set of $s \in T$ for which

(6)                    $$\|n^{-1/2}|h(s,\cdot)| - 1\|_2^2 \leq 4[c - n^{1/2}].$$

In case $c > n^{1/2}$, Chebyshev's inequality shows $\nu(M^c) \leq 1/2$, so that $\nu(M) \geq 1/2$ for any possible $c$. We now claim there are points $s_1, s_2, \ldots, s_N \in M$, with $N \leq 5^n$ and with the property that the sets

$$A_k = \{t \in T : h(s_k, t) > 7n/8\}, \qquad 1 \leq k \leq N,$$

cover $M$. To produce such points we use an $\varepsilon$-separation argument as follows.

For $s \in M$, $x(s) = (n^{-1/2}h_i(s))_{i \leq n}$ is on the unit sphere of $l_2^n$. Choose a set $B \subset M$ which is maximal with respect to the property that $\|x(s) - x(t)\|_{l_2^n} \geq 1/2$ for distinct $s, t \in B$. By [1, Lemma 2.4], $N = \operatorname{card} B \leq 5^n$. Writing $B = \{s_1, s_2, \ldots, s_N\}$, if $s \in M$ then for some $k \leq N$,

$$1/4 > \|x(s) - x(s_k)\|_{l_2^n}^2 = 2 - (2/n)h(s_k, s),$$

which means $s \in A_k$.

Finally note that $n \geq 2$ and $t \in A_k$ implies $|n^{-1/2}|h(s_k, t)| - 1|^2 \geq 1/20$. Using this and (6),

$$4[c - n^{1/2}]5^n \geq \sum_{k \leq N} \int_{A_k} |n^{-1/2}|h(s_k, \cdot)| - 1|^2 \, d\nu$$

$$\geq 1/20 \sum_{k \leq N} \nu(A_k) \geq 1/40,$$

from which $c \geq n^{1/2} + (1/5)^{n+4}$ follows.   □

PROOF OF THEOREM 1. Let $T \subset E'$ be the closed unit ball under the norm topology. By Pietsch's integral representation theorem [6, Theorem 17.3.2] there is a Borel probability $\mu$ on $T$ with

$$\|x\| \leq \pi_1(E)\mu(|\langle x, \cdot \rangle|).$$

Part (a) follows by applying Theorem 1 to the subspace $\{\langle x, \cdot \rangle | T : x \in E\} \subset C(T)$. Part (b) follows from part (a) and the following rephrasing of Theorem 2 of [4]: if $0 < \delta < 2$ and $\pi_1(E) \geq n^{1/2}(1 + \delta)$, then

$$\lambda(E) \leq n^{1/2}[1 - (2n)^{-1}(\delta/4)^2].   □$$

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