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A NOTE ON THE GENERALIZED RIEMANN INTEGRAL

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ABSTRACT. We show that the generalized Riemann integral can be defined by means of gage functions which are upper semicontinuous when restricted to a suitable subset whose complement has measure zero.

By introducing δ-fine partitions for a positive function δ (see below), Henstock and Kurzweil obtained a strikingly simple Riemannian definition of the Denjoy-Peron integral (cf. Definition 1 and [**S**, Chapter VIII]). In their definition, the function δ is completely arbitrary, and it is not clear how complicated it need be (a question of P. S. Bullen—see [**Q**]). The purpose of this note is to establish that δ can be always selected so that it is *upper semicontinuous* when restricted to a suitable subset whose complement has measure zero (cf. [$\mathbf{P_2}$, Lemma 3]). The proof is quite simple: we show first that such a δ can be chosen if the integrand is Lebesgue integrable, and then we follow the constructive Denjoy definition, observing that the upper semicontinuity property of δ is preserved at the inductive step. We also show that for a *bounded* Lebesgue integrable function, a gage δ can be selected so that it is upper semicontinuous everywhere (cf. [**FM**, Example 1]).

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By $\mathbf{R}$ and $\mathbf{R}_+$ we denote the set of all real and all positive real numbers, respectively. Unless stated otherwise, all functions in this paper are real-valued. When no confusion is possible, we denote by the same symbol a function on a set E, as well as its restrictions to various subsets of E. An interval is a compact nondegenerate subinterval of $\mathbf{R}$. A collection of intervals whose interiors are disjoint is called a *nonoverlapping* collection. If $E \subset \mathbf{R}$, then $\mathrm{cl}(E)$, $\mathrm{int}(E)$, $d(E)$, and $|E|$ denote, respectively, the closure, interior, diameter, and outer Lebesgue measure of E. A function δ on an interval A is called *nearly upper semicontinuous* if there is a set $H \subset A$ such that $|A - H| = 0$ and $\delta \restriction H$ is upper semicontinuous.

A *subpartition* of an interval A is a collection $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where $A_1, \dots, A_p$ are nonoverlapping subintervals of A, and $x_i \in A_i$, $i = 1, \dots, p$. If, in addition, $\bigcup_{i=1}^{p} A_i = A$, we say that P is a *partition* of A. Given a $\delta\colon A \to \mathbf{R}_+$, we say that a subpartition P is δ*-fine* whenever $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$. An easy compactness argument shows that a δ-fine partition of an interval A exists for each $\delta\colon A \to \mathbf{R}_+$.

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If f is a function on an interval A and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a subpartition of A, we let

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)|A_i|.$$

1. DEFINITION (HENSTOCK-KURZWEIL). A function f on an interval A is called *integrable* in A if there is a real number I with the following propety: given $\varepsilon > 0$, we can find a $\delta\colon A \to \mathbf{R}_+$ such that $|\sigma(f, P) - I| < \varepsilon$ for each δ-fine partition P of A.

Since δ-fine partitions of an interval A exist for each $\delta\colon A \to \mathbf{R}_+$, it is easy to see that the number I from the previous definition is determined uniquely by the integrable function f. It is called the *integral* of f over A, denoted by $\int_A f$, or $\int_a^b f$ if $A = [a, b]$. The family of all integrable functions on A is denoted by $\mathscr{R}(A)$.

A detailed study of the integral defined above can be found in [**H** and **K**]; an elementary exposition is given in [**Ml** and $\mathbf{P_1}$]. In particular, it is shown in [**K**, Theorem 4.14 and $\mathbf{P_1}$, Corollary B5] that the integral coincides with the Denjoy-Perron integral (see [**S**, Chapter VIII, Theorems (3.9) and (3.11)]). As this result is important for our purposes, we formulate it precisely.

The family of all *Denjoy-Perron integrable* functions on an interval A is denoted by $\mathscr{D}(A)$, and if $f \in \mathscr{D}(A)$, the symbol $(D)\int_A f$ denotes the *Denjoy-Perron integral* of f over A.

2. THEOREM (HENSTOCK-KURZWEIL). *If A is an interval, then $\mathscr{R}(A) = \mathscr{D}(A)$ and $\int_A f = (D)\int_A f$ for each $f \in \mathscr{R}(A)$.*

The function δ from Definition 1 is often referred to as a *gage* associated to f and ε. For an integrable function f on an interval A and an $\varepsilon > 0$, we denote by $\Delta(f, A; \varepsilon)$ the family of all gage functions associated to f and ε. Since positive continuous functions on compact intervals are bounded away from zero, we see immediately that f is *Riemann integrable* in the classical sense if and only if $\Delta(f, A; \varepsilon)$ contains a *continuous* gage for each $\varepsilon > 0$. Our goal is to show that for each $\varepsilon > 0$, the family $\Delta(f, A; \varepsilon)$ always contains a *nearly upper semicontinuous* gage. To this end, we denote by $\mathscr{R}_*(A)$ the family of all $f \in \mathscr{R}(A)$ such that $\Delta(f, A; \varepsilon)$ contains a nearly upper semicontinuous function for each $\varepsilon > 0$, and we show that $\mathscr{R}_*(A) = \mathscr{R}(A)$.

3. LEMMA. *Let h be a lower semicontinuous function on a set $E \subset \mathbf{R}$, let $\eta > 0$, and for each $x \in E$, let $\delta(x)$ be the supremum of all numbers $\delta \in (0, 1]$ such that $y \in E$ and $|y - x| < \delta$ implies $h(y) \geq h(x) - \eta$. Then the function $x \mapsto \delta(x)$ is upper semicontinuous on E.*

PROOF. Proceeding towards a contradiction, suppose that there is an $x \in E$ and a sequence $\{x_n\}$ in E such that $\lim x_n = x$ and $\lim \delta(x_n) > \delta(x) + \alpha$ for some $\alpha > 0$. By the definition of $\delta(x)$, there is a $y \in E$ with $|y - x| < \delta(x) + \alpha/2$ and $h(y) < h(x) - \eta$. Choose a $\beta > 0$ so that $h(y) < h(x) - \eta - \beta$, and find an x_n for which $|x - x_n| < \alpha/2$, $\delta(x_n) > \delta(x) + \alpha$, and $h(x_n) \geq h(x) - \beta$. Then

$$|y - x_n| \leq |y - x| + |x - x_n| < \delta(x) + \alpha < \delta(x_n),$$

and hence $h(y) \geq h(x_n) - \eta \geq h(x) - \eta - \beta$; a contradiction.

If E is a Lebesgue measurable subset of $\mathbf{R}$, we denote by $\mathscr{L}(E)$ the family of all functions f on E for which the *finite* Lebesgue integral $(L)\int_E f$ exists.

4. PROPOSITION. *If A is an interval, then $\mathscr{L}(A) \subset \mathscr{R}_*(A)$.*

PROOF. Let $f \in \mathscr{L}(A)$, $\varepsilon > 0$, and let $\eta = \varepsilon/(|A|+1)$. There is an uper semicontinuous function $g\colon A \to [-\infty, +\infty)$, and a lower semicontinuous function $h\colon A \to (-\infty, +\infty]$ such that $g \le f \le h$ and $(L)\int_A (h-g) < \eta$. Let $E = \{x \in A\colon h(x) < +\infty\}$, and let δ_h be the positive upper semicontinuous function on E associated to $h \restriction E$ and η according to Lemma 3. If $x \in A - E$, we select any $\delta_h(x) > 0$ so that $h(y) \ge f(x) - \eta$ for each $y \in A$ with $|y-x| < \delta_h(x)$. Since $|A-E| = 0$, we have defined a nearly upper semicontinuous function $\delta_h\colon A \to \mathbf{R}_+$. Using $-g$ instead of h, we define similarly a nearly upper semicontinuous function $\delta_g\colon A \to \mathbf{R}_+$, and set $\delta = \min(\delta_h, \delta_g)$. Now if $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ-fine partition of A, then $g(x) \le f(x_i) + \eta$ and $h(x) \ge f(x_i) - \eta$ for each $x \in A_i$, $i = 1, \ldots, p$. Thus $(L)\int_{A_i} g \le (L)\int_{A_i} f \le (L)\int_{A_i} h$, and

$$(L)\int_{A_i} g - \eta|A_i| \le f(x_i)|A_i| \le (L)\int_{A_i} h + \eta|A_i|, \qquad i = 1, \ldots, p.$$

It follows that

$$\begin{aligned}\left|\sigma(f,P) - (L)\int_A f\right| &\le \sum_{i=1}^{p} \left|f(x_i)|A_i| - (L)\int_{A_i} f\right| \\ &\le \sum_{i=1}^{p} \left[\eta|A_i| + (L)\int_{A_i}(h-g)\right] < \varepsilon,\end{aligned}$$

and we have $f \in \mathscr{R}_*(A)$.

If $f \in \mathscr{L}(A)$ then, in general, $\Delta(f, A; \varepsilon)$ may contain no gage which is a Baire functions on the whole interval A (see [**FM**, Example 1]). However, a closer look at the proof of Proposition 4 shows that $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for each $\varepsilon > 0$ whenever f has an upper semicontinuous majorant and a lower semicontinuous minorant which are both *finite*. In particular, we have the following corollary.

5. COROLLARY. *If f is a bounded Lebesgue integrable function on an interval A, then $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for every $\varepsilon > 0$.*

6. REMARK. The proofs of Proposition 4 and Corollary 5 translate verbatim to the higher dimensional Henstock-Kurzweil integrals, as well as to the integral defined by McShane in [**Ms**]. Since the McShane integral coincides with that of Lebesgue (see [$\mathbf{P_1}$, Corollary B11 or **Ml**, §8.3]), we see that it can be always defined by means of nearly upper semicontinuous gages, which can be taken upper semicontinuous whenever the integrand is bounded. This remains true even for a general setting discussed in [**AP**], provided the underlying space is metrizable.

7. LEMMA. *Let $A = [a, b]$ be an interval.*

(i) *The family $\mathscr{R}_*(A)$ is a real vector space.*

(ii) *If $f \in \mathscr{R}_*(A)$, then $f \in \mathscr{R}_*(B)$ for each subinterval B of A.*

(iii) *If $f \in \mathscr{R}_*(A)$ and $\varepsilon > 0$, then there is a nearly upper semicontinuous function $\delta\colon A \to \mathbf{R}_+$ such that*

$$\sum_{i=1}^{p} \left|f(x_i)|A_i| - \int_{A_i} f\right| < \varepsilon$$

for each δ-fine subpartition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ of A.

(iv) *If* $c \in (a,b)$ *and* $f\colon A \to \mathbf{R}$ *belongs to* $\mathscr{R}_*([a,c])$ *and* $\mathscr{R}_*([c,b])$, *then* f *belongs also to* $\mathscr{R}_*([a,b])$.

(v) *If* $f\colon A \to \mathbf{R}$ *belongs to* $\mathscr{R}_*([a,c])$ *for each* $c \in (a,b)$, *and a finite limit* $\lim_{c\to b-} \int_a^c f = I$ *exists, then* $f \in \mathscr{R}_*([a,b])$ *and* $\int_a^b f = I$.

PROOF. The proofs of properties (i)–(v) are the same as those of the corresponding properties of the Henstock-Kurzweil integral (cf. [**Ml**, §§2.1, 2.3, S3.7, 2.4, and S2.8]). We only need to observe two facts:

(1) A function which is equal almost everywhere to a nearly upper semicontinuous function is itself nearly upper semicontinuous.

(2) The distance function from a subset of $\mathbf{R}$ is continuous.

For illustration, we sketch a fairly complicated proof of property (v), following the pattern of [$\mathbf{P_1}$, Theorem A7].

Choose an $\varepsilon > 0$, and find a $\gamma \in (a,b)$ so that $|\int_a^c f - I| < \varepsilon/3$ for each $c \in [\gamma, b)$, and $|f(b)|(b-\gamma) < \varepsilon/3$. Select a strictly increasing sequence $\{c_n\}_{n=0}^\infty$ in $[a,b)$ with $c_0 = a$ and $\lim c_n = b$. By (iii), for each $n = 1,2,\ldots$, there is a nearly upper semicontinuous $\delta_n\colon [c_{n-1}, c_n] \to \mathbf{R}_+$ such that

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - \int_{A_i} f \right| < \frac{\varepsilon}{3} 2^{-n}$$

whenever $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ is a δ_n-fine subpartition of $[c_{n-1},c_n]$. In view of observations (1) and (2), we may assume that

$$\delta_n(x) \le \min(|x - c_{n-1}|, |x - c_n|)$$

for each $x \in (c_{n-1}, c_n)$, $\delta_1(c_0) \le c_1 - c_0$, and

$$\delta_n(c_n) = \delta_{n+1}(c_n) \le \min(c_n - c_{n-1}, c_{n+1} - c_n), \qquad n = 1,2,\ldots.$$

Clearly, the function δ on A defined by

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if } x \in [c_{n-1}, c_n],\ n = 1,2,\ldots, \\ b - \gamma & \text{if } x = b, \end{cases}$$

is positive and nearly upper semicontinuous. We show that it belongs to $\Delta(f, A; \varepsilon)$.

Let $P = \{(A_1,x_1),\ldots,(A_p,x_p)\}$ be a δ-fine partition of A. After a suitable reordering, we may assume that $A_i = [t_{i-1}, t_i]$, $i = 1,\ldots,p$, where $a = t_0 < \cdots < t_p = b$. Replacing (A_i, x_i) in P by $\{([t_{i-1}, x_i], x_i), ([x_i, t_i], x_i)\}$, whenever $x_i \in (t_{i-1}, t_i)$, we obtain a δ-fine partition Q of A with $\sigma(f,Q) = \sigma(f,P)$. Thus with no loss of generality we may also assume that $x_i = t_{i-1}$ or $x_i = t_i$ for each $i = 1,\ldots,p$. From this and the choice of δ, we make the following conclusion: if

$$P_n = \{(A_i, x_i) \in P\colon A_i \subset [c_{n-1}, c_n]\}$$

for $n = 1,2,\ldots$, and if N is the first positive integer with $c_N \ge t_{p-1}$, then conditions (a)–(c) below are satisfied.

(a) P_n is a δ_n-fine partition of $[c_{n-1}, c_n]$ for $n = 1,\ldots,N-1$.

(b) P_N is a δ_N-fine partition of $[c_{N-1}, t_{p-1}]$; in particular, P_N is a δ_N-fine subpartition of $[c_{N-1}, c_N]$.

(c) $P = (\bigcup_{n=1}^{N} P_n) \cup \{([t_{p-1}, b], b)\}$.

Now we have

$$|\sigma(f,P) - I| \leq \left|\sigma(f,P) - \int_a^{t_{p-1}} f\right| + \left|\int_a^{t_{p-1}} f - I\right| < \sum_{n=1}^{N-1} \left|\sigma(f,P_n) - \int_{c_{n-1}}^{c_n} f\right|$$

$$+ \left|\sigma(f,P_N) - \int_{c_{N-1}}^{t_{p-1}} f\right| + |f(b)|(b - t_{p-1}) + \frac{\varepsilon}{3} < \frac{\varepsilon}{3}\sum_{n=1}^{N} 2^{-n} + 2\cdot\frac{\varepsilon}{3} < \varepsilon,$$

and the proof is completed.

8. LEMMA. *Let f be a function on an interval A, and let $\{B_n : n = 1, 2, \ldots\}$ be a disjoint family of subintervals of A such that $f \in \mathscr{R}_*(B_n)$ for $n = 1, 2, \ldots$, and $f \in \mathscr{L}(A - \bigcup_{n\geq 1} B_n)$. Further let $W_n = \sup|\int_C f|$ where the supremum is taken over all intervals $C \subset B_n$, and suppose that $\sum_{n\geq 1} W_n < +\infty$. Then $f \in \mathscr{R}_*(A)$.*

PROOF. Let $S = A - \bigcup_{n\geq 1} B_n$ and $I = (L)\int_S f + \sum_{n\geq 1}\int_{B_n} f$. Since $|\int_{B_n} f| \leq W_n$, we see that I is a well-defined real number. Choose an $\varepsilon > 0$, and find an integer $N \geq 1$ with $\sum_{n>N} W_n < \varepsilon/6$. Let $G = \bigcup_{n>N} \operatorname{int}(B_n)$, $T = A - (G \cup \bigcup_{n=1}^{N} B_n)$, and let φ, ψ, and h be, respectively, the functions $f \restriction \bigcup_{n=1}^{N} B_n$, $f \restriction T$, and $f \restriction G$ extended to A by zero. Thus $f = \varphi + \psi + h$, and it follows from Proposition 4 and Lemma 7 (iv) that $\varphi \in \mathscr{R}_*(A)$ and $\int_A \varphi = \sum_{n=1}^{N}\int_{B_n} f$. Since T differs from S only by a countable set, Proposition 4 implies that $\psi \in \mathscr{R}_*(A)$, and we have $\int_A \psi = (L)\int_S f$. Hence by Lemma 7, (i), the function $g = \varphi + \psi$ belongs to $\mathscr{R}_*(A)$. Consequently, we can find a nearly upper semicontinuous function $\delta_g : A \to \mathbf{R}_+$ so that

$$\left|\sigma(g,P) - \left[(L)\int_S f + \sum_{n=1}^{N}\int_{B_n} f\right]\right| < \frac{\varepsilon}{3}$$

for each δ_g-fine partition P of A. By Lemma 7, (iii), there is a nearly upper semicontinuous function $\delta_n : B_n \to (0,1]$ such that

$$\sum_{i=1}^{q}\left|f(z_i)|E_i| - \int_{E_i} f\right| < \frac{\varepsilon}{3}2^{-n}$$

for each δ_n-fine subpartition $\{(E_1, z_1), \ldots, (E_q, z_q)\}$ of B_n, $n = 1, 2, \ldots$. In view of observation (2) in the proof of Lemma 7, we may assume that $(x - \delta_n(x), x + \delta_n(x)) \subset B_n$ whenever $x \in \operatorname{int}(B_n)$. We define a nearly upper semicontinuous function $\delta_h : A \to (0,1]$ by setting

$$\delta_h(x) = \begin{cases} \delta_n(x) & \text{if } x \in B_n \text{ and } n > N, \\ 1 & \text{otherwise,} \end{cases}$$

and we show that $\delta = \min(\delta_g, \delta_h)$ belongs to $\Delta(f, A; \varepsilon)$.

To this end, let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a δ-fine partition of A. For $n = 1, 2, \ldots$, denote by K_n the set of all the integers k with $1 \leq k \leq p$ and $x_k \in \operatorname{int}(B_n)$, and set $P_n = \{(A_k, x_k) \in P : k \in K_n\}$ and $C_n = \operatorname{cl}(B_n - \bigcup_{k\in K_n} A_k)$. Then each C_n is a union of at most *two* intervals, and by our choice of δ_h, for each $n > N$, the collection P_n is a δ_n-fine subpartition of B_n; in particular, $B_n = C_n \cup \bigcup_{k\in K_n} A_k$.

As $h = f$ on G and $h = 0$ on $A - G$, we have

$$\begin{aligned} |\sigma(f,P) - I| &\le \left|\sigma(g,P) - \left[(L)\int_S f + \sum_{n=1}^{N}\int_{B_n} f\right]\right| + \left|\sigma(h,P) - \sum_{n>N}\int_{B_n} f\right| \\ &< \frac{\varepsilon}{3} + \left|\sum_{n\ge N}\sum_{k\in K_n} f(x_k)|A_k| - \sum_{n\ge N}\left[\sum_{k\in K_n}\int_{A_k} f + \int_{C_n} f\right]\right| \\ &\le \frac{\varepsilon}{3} + \sum_{n\ge N}\sum_{k\in K_n}\left|f(x_k)|A_k| - \int_{A_k} f\right| + \sum_{n>N}\left|\int_{C_n} f\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\sum_{n>N} 2^{-n} + 2\sum_{n>N} W_n < \varepsilon, \end{aligned}$$

and the lemma is proved.

9. THEOREM. *If A is an interval, then $\mathscr{R}_*(A) = \mathscr{R}(A)$.*

PROOF. In view of Theorem 2, it suffices to show that $\mathscr{D}(A) \subset \mathscr{R}_*(A)$. However, by means of Proposition 4 and Lemmas 7 and 8, this follows readily from the constructive definition of the Denjoy-Perron integral (see [**S**, Chapter VIII, §5] or [**N**, Chapter XVI, §§6 and 7]).

10. REMARK. The proof of Theorem 9 does not generalize to higher dimensions (cf. Remark 6). Indeed, the proof is based on the possibility of obtaining the Henstock-Kurzweil integral by the Denjoy transfinite process, for which no satisfactory analogue in higher dimensions is known. Thus it is an open question whether Theorem 9 holds for the higher dimensional Henstock-Kurzweil integral (see [**Ml**]), or for its generalizations defined in [**M**, **JKS**, and $\mathbf{P_3}$].

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