A NOTE ON THE GENERALIZED RIEMANN INTEGRAL WASHEK F. PFEFFER

(Communicated by R. Daniel Mauldin)

Dedicated to Professor Ralph Henstock for his 65th birthday

ABSTRACT. We show that the generalized Riemann integral can be defined by means of gage functions which are upper semicontinuous when restricted to a suitable subset whose complement has measure zero.

By introducing δ -fine partitions for a positive function δ (see below), Henstock and Kurzweil obtained a strikingly simple Riemannian definition of the Denjoy-Peron integral (cf. Definition 1 and [S, Chapter VIII]). In their definition, the function δ is completely arbitrary, and it is not clear how complicated it need be (a question of P. S. Bullen—see [Q]). The purpose of this note is to establish that δ can be always selected so that it is *upper semicontinuous* when restricted to a suitable subset whose complement has measure zero (cf. [P₂, Lemma 3]). The proof is quite simple: we show first that such a δ can be chosen if the integrand is Lebesgue integrable, and then we follow the constructive Denjoy definition, observing that the upper semicontinuity property of δ is preserved at the inductive step. We also show that for a *bounded* Lebesgue integrable function, a gage δ can be selected so that it is upper semicontinuous everywhere (cf. [FM, Example 1]).

The author is obliged to J. Foran for pointing out a serious error in the preprint of this paper.

By **R** and **R**₊ we denote the set of all real and all positive real numbers, respectively. Unless stated otherwise, all functions in this paper are real-valued. When no confusion is possible, we denote by the same symbol a function on a set E, as well as its restrictions to various subsets of E. An interval is a compact nondegenerate subinterval of **R**. A collection of intervals whose interiors are disjoint is called a *nonoverlapping* collection. If $E \subset \mathbf{R}$, then cl(E), int(E), d(E), and |E| denote, respectively, the closure, interior, diameter, and outer Lebesgue measure of E. A function δ on an interval A is called *nearly upper semicontinuous* if there is a set $H \subset A$ such that |A - H| = 0 and $\delta \upharpoonright H$ is upper semicontinuous.

A subpartition of an interval A is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where A_1, \ldots, A_p are nonoverlapping subintervals of A, and $x_i \in A_i, i = 1, \ldots, p$. If, in addition, $\bigcup_{i=1}^p A_i = A$, we say that P is a partition of A. Given a $\delta \colon A \to \mathbf{R}_+$, we say that a subpartition P is δ -fine whenever $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$. An easy compactness argument shows that a δ -fine partition of an interval A exists for each $\delta \colon A \to \mathbf{R}_+$.

Received by the editors June 11, 1987.

¹⁹⁸⁰ Mathematics Subject Classification (1985 Revision). Primary 26A39.

If f is a function on an interval A and $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is a subpartition of A, we let

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i) |A_i|.$$

1. DEFINITION (HENSTOCK-KURZWEIL). A function f on an interval A is called *integrable* in A if there is a real number I with the following propety: given $\varepsilon > 0$, we can find a $\delta: A \to \mathbf{R}_+$ such that $|\sigma(f, P) - I| < \varepsilon$ for each δ -fine partition P of A.

Since δ -fine partitions of an interval A exist for each $\delta: A \to \mathbf{R}_+$, it is easy to see that the number I from the previous definition is determined uniquely by the integrable function f. It is called the *integral* of f over A, denoted by $\int_A f$, or $\int_a^b f$ if A = [a, b]. The family of all integrable functions on A is denoted by $\mathscr{R}(A)$.

A detailed study of the integral defined above can be found in [H and K]; an elementary exposition is given in [Ml and P_1]. In particular, it is shown in [K, Theorem 4.14 and P_1 , Corollary B5] that the integral coincides with the Denjoy-Perron integral (see [S, Chapter VIII, Theorems (3.9) and (3.11)]). As this result is important for our purposes, we formulate it precisely.

The family of all *Denjoy-Perron integrable* functions on an interval A is denoted by $\mathscr{D}(A)$, and if $f \in \mathscr{D}(A)$, the symbol $(D) \int_A f$ denotes the *Denjoy-Perron integral* of f over A.

2. THEOREM (HENSTOCK-KURZWEIL). If A is an interval, then $\mathscr{R}(A) = \mathscr{D}(A)$ and $\int_A f = (D) \int_A f$ for each $f \in \mathscr{R}(A)$.

The function δ from Definition 1 is often referred to as a gage associated to fand ε . For an integrable function f on an interval A and an $\varepsilon > 0$, we denote by $\Delta(f, A; \varepsilon)$ the family of all gage functions associated to f and ε . Since positive continuous functions on compact intervals are bounded away from zero, we see immediately that f is *Riemann integrable* in the classical sense if and only if $\Delta(f, A; \varepsilon)$ contains a *continuous* gage for each $\varepsilon > 0$. Our goal is to show that for each $\varepsilon > 0$, the family $\Delta(f, A; \varepsilon)$ always contains a *nearly upper semicontinuous* gage. To this end, we denote by $\mathscr{R}_*(A)$ the family of all $f \in \mathscr{R}(A)$ such that $\Delta(f, A; \varepsilon)$ contains a nearly upper semicontinuous function for each $\varepsilon > 0$, and we show that $\mathscr{R}_*(A) = \mathscr{R}(A)$.

3. LEMMA. Let h be a lower semicontinuous function on a set $E \subset \mathbf{R}$, let $\eta > 0$, and for each $x \in E$, let $\delta(x)$ be the supremum of all numbers $\delta \in (0, 1]$ such that $y \in E$ and $|y - x| < \delta$ implies $h(y) \ge h(x) - \eta$. Then the function $x \mapsto \delta(x)$ is upper semicontinuous on E.

PROOF. Proceeding towards a contradiction, suppose that there is an $x \in E$ and a sequence $\{x_n\}$ in E such that $\lim x_n = x$ and $\lim \delta(x_n) > \delta(x) + \alpha$ for some $\alpha > 0$. By the definition of $\delta(x)$, there is a $y \in E$ with $|y - x| < \delta(x) + \alpha/2$ and $h(y) < h(x) - \eta$. Choose a $\beta > 0$ so that $h(y) < h(x) - \eta - \beta$, and find an x_n for which $|x - x_n| < \alpha/2$, $\delta(x_n) > \delta(x) + \alpha$, and $h(x_n) \ge h(x) - \beta$. Then

$$|y - x_n| \le |y - x| + |x - x_n| < \delta(x) + \alpha < \delta(x_n)$$

and hence $h(y) \ge h(x_n) - \eta \ge h(x) - \eta - \beta$; a contradiction.

If E is a Lebesgue measurable subset of **R**, we denote by $\mathscr{L}(E)$ the family of all functions f on E for which the *finite* Lebesgue integral $(L) \int_E f$ exists.

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4. PROPOSITION. If A is an interval, then $\mathscr{L}(A) \subset \mathscr{R}_*(A)$.

PROOF. Let $f \in \mathscr{L}(A)$, $\varepsilon > 0$, and let $\eta = \varepsilon/(|A| + 1)$. There is an uper semicontinuous function $g: A \to [-\infty, +\infty)$, and a lower semicontinuous function $h: A \to (-\infty, +\infty]$ such that $g \leq f \leq h$ and $(L) \int_A (h-g) < \eta$. Let $E = \{x \in A: h(x) < +\infty\}$, and let δ_h be the positive upper semicontinuous function on Eassociated to $h \upharpoonright E$ and η according to Lemma 3. If $x \in A - E$, we select any $\delta_h(x) > 0$ so that $h(y) \geq f(x) - \eta$ for each $y \in A$ with $|y - x| < \delta_h(x)$. Since |A - E| = 0, we have defined a nearly upper semicontinuous function $\delta_h: A \to \mathbf{R}_+$. Using -g instead of h, we define similarly a nearly upper semicontinuous function $\delta_g: A \to \mathbf{R}_+$, and set $\delta = \min(\delta_h, \delta_g)$. Now if $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ is a δ -fine partition of A, then $g(x) \leq f(x_i) + \eta$ and $h(x) \geq f(x_i) - \eta$ for each $x \in A_i$, $i = 1, \dots, p$. Thus $(L) \int_{A_i} g \leq (L) \int_{A_i} f \leq (L) \int_{A_i} h$, and

$$(L)\int_{A_i}g-\eta|A_i|\leq f(x_i)|A_i|\leq (L)\int_{A_i}h+\eta|A_i|, \qquad i=1,\ldots,p.$$

It follows that

$$\left| \sigma(f,P) - (L) \int_{A} f \right| \leq \sum_{i=1}^{p} \left| f(x_{i}) |A_{i}| - (L) \int_{A_{i}} f \right|$$
$$\leq \sum_{i=1}^{p} \left[\eta |A_{i}| + (L) \int_{A_{i}} (h-g) \right] < \varepsilon,$$

and we have $f \in \mathscr{R}_*(A)$.

If $f \in \mathscr{L}(A)$ then, in general, $\Delta(f, A; \varepsilon)$ may contain no gage which is a Baire functions on the whole interval A (see [FM, Example 1]). However, a closer look at the proof of Proposition 4 shows that $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for each $\varepsilon > 0$ whenever f has an upper semicontinuous majorant and a lower semicontinuous minorant which are both *finite*. In particular, we have the following corollary.

5. COROLLARY. If f is a bounded Lebesgue integrable function on an interval A, then $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for every $\varepsilon > 0$.

6. REMARK. The proofs of Proposition 4 and Corollary 5 translate verbatim to the higher dimensional Henstock-Kurzweil integrals, as well as to the integral defined by McShane in [Ms]. Since the McShane integral coincides with that of Lebesgue (see [P_1 , Corollary B11 or Ml, §8.3]), we see that it can be always defined by means of nearly upper semicontinuous gages, which can be taken upper semicontinuous whenever the integrand is bounded. This remains true even for a general setting discussed in [AP], provided the underlying space is metrizable.

7. LEMMA. Let A = [a, b] be an interval.

(i) The family $\mathscr{R}_*(A)$ is a real vector space.

(ii) If $f \in \mathcal{R}_*(A)$, then $f \in \mathcal{R}_*(B)$ for each subinterval B of A.

(iii) If $f \in \mathscr{R}_*(A)$ and $\varepsilon > 0$, then there is a nearly upper semicontinuous function $\delta \colon A \to \mathbf{R}_+$ such that

$$\sum_{i=1}^{p} \left| f(x_i) |A_i| - \int_{A_i} f \right| < \varepsilon$$

for each δ -fine subpartition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ of A.

(iv) If $c \in (a,b)$ and $f: A \to \mathbf{R}$ belongs to $\mathscr{R}_*([a,c])$ and $\mathscr{R}_*([c,b])$, then f belongs also to $\mathscr{R}_*([a,b])$.

(v) If $f: A \to \mathbf{R}$ belongs to $\mathscr{R}_*([a,c])$ for each $c \in (a,b)$, and a finite limit $\lim_{c\to b^-} \int_a^c f = I$ exists, then $f \in \mathscr{R}_*([a,b])$ and $\int_a^b f = I$.

PROOF. The proofs of properties (i)-(v) are the same as those of the corresponding properties of the Henstock-Kurzweil integral (cf. [Ml, §§2.1, 2.3, S3.7, 2.4, and S2.8]). We only need to observe two facts:

(1) A function which is equal almost everywhere to a nearly upper semicontinuous function is itself nearly upper semicontinuous.

(2) The distance function from a subset of \mathbf{R} is continuous.

For illustration, we sketch a fairly complicated proof of property (v), following the pattern of $[\mathbf{P}_1, \text{Theorem A7}]$.

Choose an $\varepsilon > 0$, and find a $\gamma \in (a, b)$ so that $|f_a^c f - I| < \varepsilon/3$ for each $c \in [\gamma, b)$, and $|f(b)|(b - \gamma) < \varepsilon/3$. Select a strictly increasing sequence $\{c_n\}_{n=0}^{\infty}$ in [a, b) with $c_0 = a$ and $\lim c_n = b$. By (iii), for each $n = 1, 2, \ldots$, there is a nearly upper semicontinuous $\delta_n: [c_{n-1}, c_n] \to \mathbf{R}_+$ such that

$$\sum_{i=1}^{p} \left| f(x_i) |A_i| - \int_{A_i} f \right| < \frac{\varepsilon}{3} 2^{-n}$$

whenever $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a δ_n -fine subpartition of $[c_{n-1}, c_n]$. In view of observations (1) and (2), we may assume that

$$\delta_n(x) \le \min(|x - c_{n-1}|, |x - c_n|)$$

for each $x \in (c_{n-1}, c_n), \delta_1(c_0) \le c_1 - c_0$, and

$$\delta_n(c_n) = \delta_{n+1}(c_n) \le \min(c_n - c_{n-1}, c_{n+1} - c_n), \qquad n = 1, 2, \dots$$

Clearly, the function δ on A defined by

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if } x \in [c_{n-1}, c_n], \ n = 1, 2, \dots, \\ b - \gamma & \text{if } x = b, \end{cases}$$

is positive and nearly upper semicontinuous. We show that it belongs to $\Delta(f, A; \varepsilon)$.

Let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a δ -fine partition of A. After a suitable reordering, we may assume that $A_i = [t_{i-1}, t_i], i = 1, \ldots, p$, where $a = t_0 < \cdots < t_p = b$. Replacing (A_i, x_i) in P by $\{([t_{i-1}, x_i], x_i), ([x_i, t_i], x_i)\}$, whenever $x_i \in (t_{i-1}, t_i)$, we obtain a δ -fine partition Q of A with $\sigma(f, Q) = \sigma(f, P)$. Thus with no loss of generality we may also assume that $x_i = t_{i-1}$ or $x_i = t_i$ for each $i = 1, \ldots, p$. From this and the choice of δ , we make the following conclusion: if

$$P_{n} = \{ (A_{i}, x_{i}) \in P \colon A_{i} \subset [c_{n-1}, c_{n}] \}$$

for n = 1, 2, ..., and if N is the first positive integer with $c_N \ge t_{p-1}$, then conditions (a)-(c) below are satisfied.

(a) P_n is a δ_n -fine partition of $[c_{n-1}, c_n]$ for $n = 1, \ldots, N-1$.

(b) P_N is a δ_N -fine partition of $[c_{N-1}, t_{p-1}]$; in particular, P_N is a δ_N -fine subpartition of $[c_{N-1}, c_N]$. (c) $P = (\bigcup_{n=1}^{N} P_n) \cup \{([t_{p-1}, b], b)\}.$ Now we have

$$\begin{aligned} |\sigma(f,P) - I| &\leq \left| \sigma(f,P) - \int_{a}^{t_{p-1}} f \right| + \left| \int_{a}^{t_{p-1}} f - I \right| \\ &\leq \sum_{n=1}^{N-1} \left| \sigma(f,P_n) - \int_{c_{n-1}}^{c_n} f \right| \\ &+ \left| \sigma(f,P_N) - \int_{c_{N-1}}^{t_{p-1}} f \right| + |f(b)|(b-t_{p-1}) + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} \sum_{n=1}^{N} 2^{-n} + 2 \cdot \frac{\varepsilon}{3} \\ &\leq \varepsilon, \end{aligned}$$

and the proof is completed.

8. LEMMA. Let f be a function on an interval A, and let $\{B_n : n = 1, 2, ...\}$ be a disjoint family of subintervals of A such that $f \in \mathscr{R}_*(B_n)$ for n = 1, 2, ..., and $f \in \mathscr{L}(A - \bigcup_{n \ge 1} B_n)$. Further let $W_n = \sup |\int_C f|$ where the supremum is taken over all intervals $C \subset B_n$, and suppose that $\sum_{n \ge 1} W_n < +\infty$. Then $f \in \mathscr{R}_*(A)$.

PROOF. Let $S = A - \bigcup_{n \ge 1} B_n$ and $I = (L) \int_S f + \sum_{n \ge 1} \int_{B_n} f$. Since $|\int_{B_n} f| \le W_n$, we see that I is a well-defined real number. Choose an $\varepsilon > 0$, and find an integer $N \ge 1$ with $\sum_{n>N} W_n < \varepsilon/6$. Let $G = \bigcup_{n>N} \operatorname{int}(B_n)$, $T = A - (G \cup \bigcup_{n=1}^N B_n)$, and let φ , ψ , and h be, respectively, the functions $f \upharpoonright \bigcup_{n=1}^N B_n$, $f \upharpoonright T$, and $f \upharpoonright G$ extended to A by zero. Thus $f = \varphi + \psi + h$, and it follows from Proposition 4 and Lemma 7 (iv) that $\varphi \in \mathscr{R}_*(A)$ and $\int_A \varphi = \sum_{n=1}^N \int_{B_n} f$. Since T differs from S only by a countable set, Proposition 4 implies that $\psi \in \mathscr{R}_*(A)$, and we have $\int_A \psi = (L) \int_S f$. Hence by Lemma 7, (i), the function $g = \varphi + \psi$ belongs to $\mathscr{R}_*(A)$. Consequently, we can find a nearly upper semicontinuous function $\delta_g \colon A \to \mathbf{R}_+$ so that

$$\left|\sigma(g,P) - \left[(L)\int_{S}f + \sum_{n=1}^{N}\int_{B_{n}}f\right]\right| < \frac{\varepsilon}{3}$$

for each δ_g -fine partition P of A. By Lemma 7, (iii), there is a nearly upper semicontinuous function $\delta_n: B_n \to (0, 1]$ such that

$$\sum_{i=1}^{q} \left| f(z_i) |E_i| - \int_{E_i} f \right| < \frac{\varepsilon}{3} 2^{-n}$$

for each δ_n -fine subpartition $\{(E_1, z_1), \ldots, (E_q, z_q)\}$ of B_n , $n = 1, 2, \ldots$ In view of observation (2) in the proof of Lemma 7, we may assume that $(x - \delta_n(x), x + \delta_n(x)) \subset B_n$ whenever $x \in int(B_n)$. We define a nearly upper semicontinuous function $\delta_h \colon A \to (0, 1]$ by setting

$$\delta_h(x) = \begin{cases} \delta_n(x) & \text{if } x \in B_n \text{ and } n > N, \\ 1 & \text{otherwise,} \end{cases}$$

and we show that $\delta = \min(\delta_g, \delta_h)$ belongs to $\Delta(f, A; \varepsilon)$.

To this end, let $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a δ -fine partition of A. For $n = 1, 2, \ldots$, denote by K_n the set of all the integers k with $1 \le k \le p$ and $x_k \in int(B_n)$, and set $P_n = \{(A_k, x_k) \in P : k \in K_n\}$ and $C_n = cl(B_n - \bigcup_{k \in K_n} A_k)$. Then each C_n is a union of at most *two* intervals, and by our choice of δ_h , for each n > N, the collection P_n is a δ_n -fine subpartition of B_n ; in particular, $B_n = C_n \cup \bigcup_{k \in K_n} A_k$.

As h = f on G and h = 0 on A - G, we have

$$\begin{aligned} |\sigma(f,P) - I| &\leq \left| \sigma(g,P) - \left[(L) \int_{S} f + \sum_{n=1}^{N} \int_{B_{n}} f \right] \right| + \left| \sigma(h,P) - \sum_{n > N} \int_{B_{n}} f \right| \\ &\leq \frac{\varepsilon}{3} + \left| \sum_{n \geq N} \sum_{k \in K_{n}} f(x_{k}) |A_{k}| - \sum_{n \geq N} \left[\sum_{k \in K_{n}} \int_{A_{k}} f + \int_{C_{n}} f \right] \right| \\ &\leq \frac{\varepsilon}{3} + \sum_{n \geq N} \sum_{k \in K_{n}} \left| f(x_{k}) |A_{k}| - \int_{A_{k}} f \right| + \sum_{n > N} \left| \int_{C_{n}} f \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \sum_{n > N} 2^{-n} + 2 \sum_{n > N} W_{n} < \varepsilon, \end{aligned}$$

and the lemma is proved.

9. THEOREM. If A is an interval, then $\mathscr{R}_*(A) = \mathscr{R}(A)$.

PROOF. In view of Theorem 2, it suffices to show that $\mathscr{D}(A) \subset \mathscr{R}_*(A)$. However, by means of Proposition 4 and Lemmas 7 and 8, this follows readily from the constructive definition of the Denjoy-Perron integral (see [S, Chapter VIII, §5] or [N, Chapter XVI, §§6 and 7]).

10. REMARK. The proof of Theorem 9 does not generalize to higher dimensions (cf. Remark 6). Indeed, the proof is based on the possibility of obtaining the Henstock-Kurzweil integral by the Denjoy transfinite process, for which no satisfactory analogue in higher dimensions is known. Thus it is an open question whether Theorem 9 holds for the higher dimensional Henstock-Kurzweil integral (see [MI]), or for its generalizations defined in [M, JKS, and P₃].

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