

## A NOTE ON EXTREME POINTS OF SUBORDINATION CLASSES

D. J. HALLENBECK

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**ABSTRACT.** Let  $s(F)$  denote the set of functions subordinate to a univalent function  $F$  in  $\Delta$  in the unit disc. Let  $B$  denote the set of functions  $\phi(z)$  analytic in  $\Delta$  satisfying  $|\phi(x)| < 1$  and  $\phi(0) = 0$ . Let  $D = F(\Delta)$  and  $\lambda(w, \partial D)$  denote the distance between  $w$  and  $\partial D$  (boundary of  $D$ ). We prove that if  $\phi$  is an extreme point of  $B$  then  $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$ . As a corollary we prove that if  $F \circ \phi$  is an extreme point of  $s(F)$  then  $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$ .

**Introduction.** Let  $\Delta = \{z: |z| < 1\}$  and let  $\mathbf{A}$  denote the set of functions analytic in  $\Delta$ . Let  $B$  denote the subset of  $\mathbf{A}$  consisting of all functions  $\phi$  that satisfy the conditions  $|\phi(z)| < 1$ ,  $\phi(0) = 0$ . Let  $EB$  denote the extreme points of  $B$ . Let  $S$  denote the subset of  $\mathbf{A}$  consisting of univalent functions  $f$  so that  $f(z) = z + \dots$  in  $\Delta$ .

Let  $F$  be in  $\mathbf{A}$  and be univalent in  $\Delta$ . Let  $s(F)$  denote the subset of  $\mathbf{A}$  consisting of functions  $f$  that are subordinate to  $F$  in  $\Delta$ . This means that  $f \in \mathbf{A}$ ,  $f(0) = F(0)$ , and  $f(\Delta) \subset F(\Delta)$ . These conditions are equivalent to the existence of  $\phi \in B$  so that  $f = F \circ \phi$ . Note that  $s(F) = \{F \circ \phi: \phi \in B\}$ .

Let  $D$  denote  $F(\Delta)$ . It is known that  $F \in H^p$  for all  $p < 1/2$  [4, p. 50] and so if  $f = F \circ \phi$  for  $\phi \in B$  then  $f \in H^p$  for all  $p < 1/2$  [4, pp. 10–11]. It follows that  $\lim_{r \rightarrow 1} f(re^{it}) = f(e^{it})$  exists almost everywhere. In [7] it was proved that  $f(e^{it}) = F(\phi(e^{it}))$  for almost all  $\theta$ . We let  $Es(F)$  denote the set of extreme points of  $s(F)$ . In [1] it was proved that if  $F'$  is in the Nevanlinna class and  $\phi \in EB$  then  $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$ . It was conjectured in [1] that the integral was  $-\infty$  for any univalent function  $F$  when  $\phi \in EB$ . (Note that this is trivially true if  $|\phi(e^{it})| = 1$  on a set of positive measure since  $F$  is univalent.) A weaker conjecture is that if  $F \circ \phi \in Es(F)$  and  $F$  is univalent then  $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$ . In a recent paper [2] it was proved that if  $F$  is univalent and  $\phi \in EB$  then

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt = -\infty$$

for almost all  $\theta$ . The analogous form of the weaker conjecture formulated above was also proved in [2].

In this paper we prove both conjectures.

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**Functions subordinate to a univalent function.**

**THEOREM 1.** *If  $F$  is a bounded univalent function analytic in  $\Delta$ ,  $\phi \in B$  and  $|\phi(e^{it})| < 1$  for almost all  $t$ , then*

$$(1) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))| dt < +\infty.$$

**PROOF.** Let  $g(z) = \int_0^z (F'(\tau))^2 d\tau$  where  $z = re^{i\theta}$  and  $\tau = pe^{i\theta}$  ( $0 \leq p \leq r$ ). Then  $g(z) = \int_0^r (F'(pe^{i\theta}))^2 ie^{i\theta} dp$ . Since  $F$  is a bounded univalent function,  $F(\Delta)$  has finite area. Hence,

$$(2) \quad \int_0^{2\pi} |g(re^{i\theta})| d\theta \leq \int_0^{2\pi} \left( \int_0^r |F'(pe^{i\theta})|^2 dp \right) d\theta < +\infty$$

It follows from (2) that  $g \in H^1$  and so [4, p. 2] we have by analytic completion

$$(3) \quad g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + i\beta$$

where  $\mu(t)$  is a function of bounded variation on  $[0, 2\pi]$  and  $\beta$  is a real constant. Since  $g'(z) = (F'(z))^2$  it follows from (3) that

$$(4) \quad (F'(z))^2 = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2} d\mu(t).$$

We deduce from (4) that

$$(5) \quad (1 - |z|^2) |F'(z)|^2 \leq \frac{1}{\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} |d\mu(t)|.$$

Denote the right-hand side of (5) by  $u(z)$ . Since

$$w(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} |d\mu(t)|$$

is analytic in  $\Delta$  and  $u(z) = \text{Re } w(z)$  we conclude that  $u(z)$  is harmonic in  $\Delta$ .

The function  $(1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))|^2$  is positive and measurable since  $|\phi(e^{it})| < 1$  for almost all  $t$  and  $(1 - |\phi(re^{it})|^2) |F'(\phi(re^{it}))|^2$  is continuous. It follows from Fatou's lemma that

$$(6) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))|^2 dt \leq \liminf_{r \rightarrow 1} \int_0^{2\pi} (1 - |\phi(re^{it})|^2) |F'(\phi(re^{it}))|^2 dt.$$

We conclude from (5) and (6) that

$$(7) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))|^2 dt \leq \liminf_{r \rightarrow 1} \int_0^{2\pi} u(\phi(re^{it})) dt.$$

Since  $u(\phi(z))$  is harmonic in  $\Delta$  and  $\phi(0) = 0$ , the right-hand side of (7) is equal to  $2\pi u(0)$ . Hence,

$$(8) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))|^2 dt < +\infty.$$

We note that (1) follows from (8) by an application of the Cauchy-Schwarz inequality. This completes the proof.

We next prove our main theorem.

**THEOREM 2.** *If  $F$  is a univalent function analytic in  $\Delta$ ,  $\phi \in EB$ , then*

$$(9) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty.$$

**PROOF.** We first note that by arguments given in detail in [2] it is sufficient to consider the case that  $F$  is bounded. Since (9) is easily seen to hold with  $|\phi(e^{it})| = 1$  on a set of positive measure we only consider the case  $|\phi(e^{it})| < 1$  for almost all  $t$ . By Theorem (1) we know that (1) holds. It is easy to deduce from (1) that

$$(10) \quad \int_0^{2\pi} \log[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))|] dt < +\infty.$$

Since  $F$  is univalent, it follows from [6, p. 22] that

$$(11) \quad \lambda(F(\phi(e^{it})), \partial D) \leq (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))|$$

for almost all  $t$ . It follows from (11) that we have

$$(12) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt \leq \frac{1}{2} \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt + \int_0^{2\pi} \log[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))|] dt.$$

Since  $\phi \in EB$  we have  $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt = -\infty$  [4, p. 125] and so (9) follows from this fact, (10) and (12). This completes the proof.

**THEOREM 3.** *If  $F$  is a univalent function analytic in  $\Delta$ ,  $\phi \in B$  and  $F \circ \phi \in Es(F)$  then*

$$(13) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty.$$

**PROOF.** This follows from Theorem 2 above and Theorem 1 in [2] where it was proved that if  $F \circ \phi \in Es(F)$  then  $\phi \in EB$ .

**REMARK.** Condition (13) is seen to be a necessary condition for  $F \circ \phi \in Es(F)$ . Also, it is known that  $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$  does not in general imply that  $F \circ \phi \in Es(F)$ . This can be easily seen by considering the case  $F(z) = ((1 + z)/(1 - z))^\alpha$  for  $0 < \alpha \leq 1$  [4, pp. 131, 133].

**THEOREM 4.** *Suppose  $F$  is a univalent function analytic in  $\Delta$  and  $\phi \in B$ . Then*

$$(14) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$$

*if and only if  $\phi \in EB$ .*

**PROOF.** If  $\phi \in EB$  then (14) follows from Theorem 3. The other implication was proved in [5].

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK,  
DELAWARE 19716