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A NOTE ON EXTREME POINTS OF SUBORDINATION CLASSES

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ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a univalent function F in Δ in the unit disc. Let B denote the set of functions $\phi(z)$ analytic in Δ satisfying $|\phi(x)| < 1$ and $\phi(0) = 0$. Let $D = F(\Delta)$ and $\lambda(w, \partial D)$ denote the distance between w and ∂D (boundary of D). We prove that if ϕ is an extreme point of B then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$. As a corollary we prove that if $F \circ \phi$ is an extreme point of $s(F)$ then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$.

Introduction. Let $\Delta = \{z: |z| < 1\}$ and let $\mathbf{A}$ denote the set of functions analytic in Δ. Let B denote the subset of $\mathbf{A}$ consisting of all functions ϕ that satisfy the conditions $|\phi(z)| < 1$, $\phi(0) = 0$. Let EB denote the extreme points of B. Let S denote the subset of $\mathbf{A}$ consisting of univalent functions f so that $f(z) = z + \cdots$ in Δ.

Let F be in $\mathbf{A}$ and be univalent in Δ. Let $s(F)$ denote the subset of $\mathbf{A}$ consisting of functions f that are subordinate to F in Δ. This means that $f \in \mathbf{A}$, $f(0) = F(0)$, and $f(\Delta) \subset F(\Delta)$. These conditions are equivalent to the existence of $\phi \in B$ so that $f = F \circ \phi$. Note that $s(F) = \{F \circ \phi: \phi \in B\}$.

Let D denote $F(\Delta)$. It is known that $F \in H^p$ for all $p < 1/2$ [**4**, p. 50] and so if $f = F \circ \phi$ for $\phi \in B$ then $f \in H^p$ for all $p < 1/2$ [**4**, pp. 10–11]. It follows that $\lim_{r \to 1} f(re^{it}) = f(e^{it})$ exists almost everywhere. In [**7**] it was proved that $f(e^{it}) = F(\phi(e^{it}))$ for almost all θ. We let $Es(F)$ denote the set of extreme points of $s(F)$. In [**1**] it was proved that if F' is in the Nevanlinna class and $\phi \in EB$ then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$. It was conjectured in [**1**] that the integral was $-\infty$ for any univalent function F when $\phi \in EB$. (Note that this is trivially true if $|\phi(e^{it})| = 1$ on a set of positive measure since F is univalent.) A weaker conjecture is that if $F \circ \phi \in Es(F)$ and F is univalent then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$. In a recent paper [**2**] it was proved that if F is univalent and $\phi \in EB$ then

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt = -\infty$$

for almost all θ. The analogous form of the weaker conjecture formulated above was also proved in [**2**].

In this paper we prove both conjectures.

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Functions subordinate to a univalent function.

THEOREM 1. *If F is a bounded univalent function analytic in Δ, $\phi \in B$ and $|\phi(e^{it})| < 1$ for almost all t, then*

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))| \, dt < +\infty. \tag{1}$$

PROOF. Let $g(z) = \int_0^z (F'(\tau))^2 \, d\tau$ where $z = re^{i\theta}$ and $\tau = pe^{i\theta}$ $(0 \le p \le r)$. Then $g(z) = \int_0^r (F'(pe^{i\theta}))^2 ie^{i\theta} \, dp$. Since F is a bounded univalent function, $F(\Delta)$ has finite area. Hence,

$$\int_0^{2\pi} |g(re^{i\theta})| \, d\theta \le \int_0^{2\pi} \left(\int_0^r |F'(pe^{i\theta})|^2 \, dp \right) d\theta < +\infty \tag{2}$$

It follows from (2) that $g \in H^1$ and so [**4**, p. 2] we have by analytic completion

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) + i\beta \tag{3}$$

where $\mu(t)$ is a function of bounded variation on $[0, 2\pi]$ and β is a real constant. Since $g'(z) = (F'(z))^2$ it follows from (3) that

$$(F'(z))^2 = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - z)^2} \, d\mu(t). \tag{4}$$

We deduce from (4) that

$$(1 - |z|^2)|F'(z)|^2 \le \frac{1}{\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} |\, d\mu(t)|. \tag{5}$$

Denote the right-hand side of (5) by $u(z)$. Since

$$w(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} |\, d\mu(t)|$$

is analytic in Δ and $u(z) = \operatorname{Re} w(z)$ we conclude that $u(z)$ is harmonic in Δ.

The function $(1 - |\phi(e^{it})|^2)|F'(\phi(e^{it}))|^2$ is positive and measurable since $|\phi(e^{it})| < 1$ for almost all t and $(1 - |\phi(re^{it})|^2)|F'(\phi(re^{it}))|^2$ is continuous. It follows from Fatou's lemma that

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)|F'(\phi(e^{it}))|^2 \, dt \le \underline{\lim}_{r \to 1} \int_0^{2\pi} (1 - |\phi(re^{it})|^2)|F'(\phi(re^{it}))|^2 \, dt. \tag{6}$$

We conclude from (5) and (6) that

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)|F'(\phi(e^{it}))|^2 \, dt \le \underline{\lim}_{r \to 1} \int_0^{2\pi} u(\phi(re^{it})) \, dt. \tag{7}$$

Since $u(\phi(z))$ is harmonic in Δ and $\phi(0) = 0$, the right-hand side of (7) is equal to $2\pi u(0)$. Hence,

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)|F'(\phi(e^{it}))|^2 \, dt < +\infty. \tag{8}$$

We note that (1) follows from (8) by an application of the Cauchy-Schwarz inequality. This completes the proof.

We next prove our main theorem.

THEOREM 2. *If F is a univalenvt function analytic in Δ, $\phi \in EB$, then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty. \tag{9}$$

PROOF. We first note that by arguments given in detail in [**2**] it is sufficient to consider the case that F is bounded. Since (9) is easily seen to hold with $|\phi(e^{it})| = 1$ on a set of positive measure we only consider the case $|\phi(e^{it})| < 1$ for almost all t. By Theorem (1) we know that (1) holds. It is easy to deduce from (1) that

$$\int_0^{2\pi} \log[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))|]\, dt < +\infty. \tag{10}$$

Since F is univalent, it follows from [**6**, p. 22] that

$$\lambda(F(\phi(e^{it})), \partial D) \le (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it}))| \tag{11}$$

for almost all t. It follows from (11) that we have

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt \le \frac{1}{2} \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\, dt + \int_0^{2\pi} \log[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it}))|]\, dt. \tag{12}$$

Since $\phi \in EB$ we have $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\, dt = -\infty$ [**4**, p. 125] and so (9) follows from this fact, (10) and (12). This completes the proof.

THEOREM 3. *If F is a univalent function analytic in Δ, $\phi \in B$ and $F \circ \phi \in Es(F)$ then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty. \tag{13}$$

PROOF. This follows from Theorem 2 above and Theorem 1 in [**2**] where it was proved that if $F \circ \phi \in Es(F)$ then $\phi \in EB$.

REMARK. Condition (13) is seen to be a necessary condition for $F \circ \phi \in Es(F)$. Also, it is known that $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$ does not in general imply that $F \circ \phi \in Es(F)$. This can be easily seen by considering the case $F(z) = ((1+z)/(1-z))^\alpha$ for $0 < \alpha \le 1$ [**4**, pp. 131, 133].

THEOREM 4. *Suppose F is a univalent function analytic in Δ and $\phi \in B$. Then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty \tag{14}$$

if and only if $\phi \in EB$.

PROOF. If $\phi \in EB$ then (14) follows from Theorem 3. The other implication was proved in [**5**].

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