PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

AN EXTENSION THEOREM FOR NORMAL FUNCTIONS

PENTTI JÄRVI

(Communicated by Irwin Kra)

ABSTRACT. Given a domain $\Omega \subset \mathbf{C}^n$, an analytic subvariety V of Ω and a normal function $f\colon \Omega \backslash V \to \widehat{\mathbf{C}}$, we show that f can be extended to a holomorphic mapping $f^*\colon \Omega \to \widehat{\mathbf{C}}$ provided the singularities of V are normal crossings.

1. As an extension of the big Picard theorem, Lehto and Virtanen showed [**5**, Theorem 9] that isolated singularities are removable for normal meromorphic functions. It is the purpose of this note to give a generalization of this result for functions defined in subdomains of $\mathbf{C}^n$. It is conceivable that the notion of normality can be generalized in various ways to higher dimensions. Here we adopt the definition of Cima and Krantz [**1**, p. 305].

Let $D \subset \mathbf{C}$ be the open unit disc, and let $\Omega \subset \mathbf{C}^n$ be a domain. The infinitesimal form of the Kobayashi metric for Ω at $z \in \Omega$ in the direction $\xi \in \mathbf{C}^n$ is defined to be

$$F_\Omega(z,\xi) = \inf\Big\{|\xi|/|f'(0)| \mid f\colon D \to \Omega \text{ holomorphic, } f(0) = z, \\ \text{and } f'(0) \text{ is a positive multiple of } \xi\Big\}.$$

Here $|\ |$ stands for the Euclidean length. Further, let $z \in \widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and let $\eta \in \mathbf{C}$ be thought of as a tangent vector to $\widehat{\mathbf{C}}$ at z. Then the infinitesimal form of the spherical metric at z is defined by

$$|\eta|_{\mathrm{sph},z} = |\eta|/(1+|z|^2).$$

Now suppose that f is a holomorphic mapping of Ω into $\widehat{\mathbf{C}}$, and let f' denote the matrix $(\partial f/\partial z_i)$. Then f is said to be *normal* provided there exists a constant C such that

$$(1)\qquad |f'(z)\cdot\xi|_{\mathrm{sph},f(z)} \le C\cdot F_\Omega(z,\xi) \quad \text{for all } z \in \Omega \text{ and all } \xi \in \mathbf{C}^n.$$

The minimum of those constants C, for which (1) holds true, is called the *order of normality* of f and denoted by C_f.

REMARK 1. Suppose f is a holomorphic mapping of Ω into $\widehat{\mathbf{C}}$ such that $\widehat{\mathbf{C}}\backslash f(\Omega)$ contains three points a_1, a_2 and a_3. Then f is normal. First, the distance-decreasing property of the Kobayashi metric yields

$$(2)\qquad F_{\widehat{\mathbf{C}}\backslash\{a_1,a_2,a_3\}}(f(z), f'(z)\cdot\xi) \le F_\Omega(z,\xi) \quad \text{for all } z \in \Omega \text{ and all } \xi \in \mathbf{C}^n.$$

Received by the editors May 8, 1987 and, in revised form, July 2, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32H25; Secondary 32H20.
Key words and phrases. Normal function, Kobayashi metric.

Further, making use of the homogeneity and the continuity of $F_{\widehat{\mathbf{C}}\setminus\{a_1,a_2,a_3\}}$ and the spherical metric as well as the fact that

$$|1|_{\mathrm{sph},z}/F_{\widehat{\mathbf{C}}\setminus\{a_1,a_2,a_3\}}(z,1)\to 0 \quad \text{as } z\to a_j,\ j=1,2,3,$$

one readily finds a constant C such that

$$(3)\qquad |\eta|_{\mathrm{sph},z}\le C\cdot F_{\widehat{\mathbf{C}}\setminus\{a_1,a_2,a_3\}}(z,\eta)\quad \text{for all } z\in\widehat{\mathbf{C}}\setminus\{a_1,a_2,a_3\} \text{ and all } \eta\in\mathbf{C}.$$

Combining (2) and (3) gives the assertion. Another deduction of this result is given in [**1**, p. 308].

2. Set $D^*=D\setminus\{0\}$, and let $f\colon D^*\to\widehat{\mathbf{C}}$ be normal. As noted before, f extends to a function f^* meromorphic in D [**5**, p. 62]. Since $F_D(z,\eta)$ and $F_{D^*}(z,\eta)$ are comparable near ∂D, f^* is normal in D. Moreover, the order of normality of f^* does not deviate too much from that of f. More precisely, we have

LEMMA 1. *Given a positive number K, there is a positive number K' such that any function f normal in D^* with $C_f\le K$ extends to a function f^* normal in D with $C_{f^*}\le K'$.*

PROOF. We begin with a quick proof of the Lehto-Virtanen extension theorem. Let f be normal in D^*. Recall that

$$F_{D^*}(z,\xi)=\text{hyperbolic metric of } D^*=\frac{|\xi|}{|z|\log(1/|z|)}.$$

Hence the hyperbolic area of $D^*(r)=\{z\in\mathbf{C}|0<|z|<r\}$ is finite for every $r<1$. By (1), we have

$$\iint_{D^*(r)}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,dx\,dy<\infty.$$

By the big Picard theorem, the singularity at 0 is inessential.

Fix $K>0$, and let $f\colon D^*\to\widehat{\mathbf{C}}$ be normal with $C_f\le K$. Again, let f^* stand for the extended function. Since the spherical metric is invariant under the rotations of the sphere, we may assume that $f^*(0)=0$. Clearly, it is sufficient to exhibit an R, $0<R<1$, depending only on K, such that $|f^*(z)|<1$ for $z\in D(R)=\{z\in\mathbf{C}||z|<R\}$. By assumption,

$$(4)\qquad \frac{|f'(z)|}{1+|f(z)|^2}\le K\cdot\frac{1}{|z|\log(1/|z|)}\quad \text{for all } z\in D^*.$$

Set $\gamma_r=\{z\in\mathbf{C}|\,|z|=r\}$, $0<r<1$, and let $s(f^*(\gamma_r))$ denote the spherical length of $f^*(\gamma_r)$. We claim that $R=e^{-8K}$ does the job. Suppose, on the contrary, that $|f^*(z)|\ge 1$ for some $z\in D(R)$. Pick out $z_0\in D(R)$ such that $|f^*(z_0)|=1$ and $|f^*(z)|<1$ for $|z|<|z_0|$. A simple estimate based on (4) gives $s(f^*(\gamma_{|z_0|}))<\pi/4$. Since the spherical distance of 0 and $f(z_0)$ is $\pi/2$, $f^*(\gamma_{|z_0|})$ lies in the half plane $\operatorname{Re}\overline{f(z_0)}z>0$. Therefore, the winding number of $f^*(\gamma_{|z_0|})$ with respect to 0 (in $\mathbf{C}$) is 0. This contradiction with $f^*(0)=0$ completes the proof. □

The next lemma is readily deduced by elementary considerations on the Kobayashi metric. We omit the proof.

LEMMA 2. *Let* $f\colon (D^*)^n \to \widehat{\mathbf{C}}$ *be normal.*

(1) *For every* $k \in \{1,\dots,n\}$ *and every* $(a_1,\dots,a_{n-1}) \in (D^*)^{n-1}$, *the map* $z \mapsto f(a_1,\dots,a_{k-1},z,a_k,\dots,a_{n-1})$, $D^* \to \widehat{\mathbf{C}}$, *is a normal function.*

(2) *For every* $k \in \{1,\dots,n\}$ *and every* $a \in D^*$ *the map* $(z_1,\dots,z_{n-1}) \mapsto f(z_1,\dots,z_{k-1},a,z_k,\dots,z_{n-1})$, $(D^*)^{n-1} \to \widehat{\mathbf{C}}$, *is a normal function.*

(3) *For every* $a = (a_1,\dots,a_n) \in \partial D^n$ *with* $a_i \neq 0$, $i = 1,\dots,n$, *the map* $z \mapsto f(a_1 z,\dots,a_n z)$, $D^* \to \widehat{\mathbf{C}}$, *is a normal function.*

Further, the orders of normality of all these functions are bounded above by that of f.

3. Let $\Omega \in \mathbf{C}^n$ be a domain and let $V \subset \Omega$ be an analytic subvariety of codimension one. The singularities of V are said to be *normal crossings* provided $\Omega \backslash V$ is locally biholomorphic to $(D^*)^k \times D^{n-k}$ for some $k \in \{0,\dots,n\}$. Our main theorem reads as follows.

THEOREM 1. *Let* $\Omega \subset \mathbf{C}^n$ *be a domain and let* $V \subset \Omega$ *be an analytic subvariety of codimension one, whose singularities are normal crossings. Suppose* $f\colon \Omega\backslash V \to \widehat{\mathbf{C}}$ *is normal. Then* f *extends to a holomorphic mapping* $f^*\colon \Omega \to \widehat{\mathbf{C}}$.

PROOF. Since the problem is of a local nature and the inclusion mapping is distance-decreasing (in the Kobayashi metrics), we may assume that $\Omega = D^n$ and $D^n \backslash V \doteq (D^*)^n$.

The proof will be by induction on n. The case $n = 1$ is part of Lemma 1. So let $n \geq 2$ and assume the extension is possible for $1,\dots,n-1$. Let $a = (a_1,\dots,a_n) \in V\backslash\{0\}$, and choose $k \in \{1,\dots,n\}$ such that $a_k \neq 0$. Consider the mapping $f_{a_k}\colon (z_1,\dots,z_{n-1}) \mapsto f(z_1,\dots,z_{k-1},a_k,z_k,\dots,z_{n-1})$, $(D^*)^{n-1} \to \widehat{\mathbf{C}}$. It follows from Lemma 2 that f_{a_k} is normal. Hence, by the induction hypothesis, f_{a_k} admits a holomorphic extension $f^*_{a_k}\colon D^{n-1} \to \widehat{\mathbf{C}}$. We set $f^*(a) = f^*_{a_k}(a_1,\dots,a_{k-1},a_{k+1},\dots,a_n)$. We will show that the extended mapping is holomorphic on $D^n\backslash\{0\}$. By the Riemann extension theorem, it suffices to prove that f^* is continuous, i.e., $\mathrm{Cl}(f;a)$, the cluster set of f at any $a \in V\backslash\{0\}$ reduces to a singleton (of course, this also shows that the extension does not depend on the choice of k).

So let a, k and $f^*(a)$ be as above. Pick $\varepsilon > 0$. Set

$$\chi(z,w) = \frac{|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}} \quad \text{for } z,w \in \widehat{\mathbf{C}}$$

and $B(a,\delta) = \{z \in \mathbf{C}^n \mid |z-a| < \delta\}$. By Lemma 2, the family

$$\{z \mapsto f(b_1,\dots,b_{k-1},z,b_k,\dots,b_{n-1}) \mid (b_1,\dots,b_{n-1}) \in (D^*)^{n-1}\}$$

is equicontinuous at a_k. Hence there exists a positive δ such that

$$\chi(f(z), f(z_1,\dots,z_{k-1},a_k,z_{k+1},\dots,z_n)) < \varepsilon/2$$

and

$$\chi(f(z_1,\dots,z_{k-1},a_k,z_{k+1},\dots,z_n), f^*(a)) < \varepsilon/2 \text{ for } z = (z_1,\dots,z_n) \in B(a,\delta)\backslash V.$$

Thus $\chi(f(z), f^*(a)) < \varepsilon$ for $z \in B(a,\delta)\backslash V$. It follows that $\mathrm{Cl}(f;a) = f^*(a)$.

It remains to extend f to 0. First, we infer from Lemmas 2 and 1 that $f(z,\dots,z)$ tends to a limit, say w_0, as $z \to 0$. It suffices to show that $\mathrm{Cl}(f;0) = w_0$. Let $\varepsilon > 0$.

Consider the mappings described in Lemma 2 (3), or rather their counterparts for the restrictions of f to the hyperplanes of the form $\{z_n = a\}$, $a \in D^*$. By Lemma 1, all of them extend holomorphically to 0. Moreover, it follows from Lemmas 2 and 1 that the extensions constitute an equicontinuous family at 0 ($\in \mathbf{C}$). Therefore, we find a positive δ such that $\chi(f(z), f^*(0, \ldots, 0, z_n)) < \varepsilon/3$, $\chi(f^*(0, \ldots, 0, z_n), f(z_n, \ldots, z_n)) < \varepsilon/3$ and $\chi(f(z_n, \ldots, z_n), w_0) < \varepsilon/3$ for $z = (z_1, \ldots, z_n) \in B(0, \delta)\backslash V$. Therefore $\chi(f(z), w_0) < \varepsilon$ for $z \in B(0, \delta)\backslash V$. Hence $\mathrm{Cl}(f; 0) = w_0$. This completes the proof. □

REMARK 2. Set $V = \{(z_1, z_2) \in \mathbf{C}^2 | z_1 z_2 (z_1 - z_2) = 0\}$ and consider the mapping $f\colon D^2\backslash V \to \widehat{\mathbf{C}}$, $(z_1, z_2) \mapsto z_1/z_2$. Since f omits the values 0, 1 and ∞ in $D^2\backslash V$, it is normal by Remark 1. Yet f does not extend to a holomorphic map $D^2 \to \widehat{\mathbf{C}}$. Accordingly, we cannot dispense with some restrictions on the singularities of V in Theorem 1.

REMARK 3. One may ask whether the extended function is normal in Ω (provided Ω is hyperbolic). However, it seems that this need not be the case even in dimension one.

REMARK 4. Results related to Theorem 1 can be found in [**2**, **3** and **4**]. Cf. in particular [**2**, Theorem 2].

In the counterexample discussed above the function involved is the restriction to $D^2\backslash V$ of a meromorphic function, i.e., a function with "indeterminacies". This is always the case as shown by

THEOREM 2. *Let $\Omega \subset \mathbf{C}^n$ be a domain and let V be a subvariety of Ω. Suppose $f\colon \Omega\backslash V \to \widehat{\mathbf{C}}$ is normal. Then f extends to a meromorphic function in Ω.*

PROOF. Denote by $S(V)$ the set of singular points of V. By Theorem 1 f extends to a holomorphic mapping of $\Omega\backslash S(V)$ into $\widehat{\mathbf{C}}$. Thus f can be regarded as a meromorphic function in $\Omega\backslash S(V)$. Since $\dim S(V) \leq n - 2$ [**6**, p. 144], f extends to a function meromorphic in Ω [**6**, p. 149]. □

REFERENCES

1. J. A. Cima and S. G. Krantz, *The Lindelöf principle and normal functions of several complex variables*, Duke Math. J. **50** (1983), 303–328.
2. P. Kiernan, *Extensions of holomorphic maps*, Trans. Amer. Math. Soc. **172** (1972), 347–355.
3. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure Appl. Math., no. 2, Dekker, New York, 1970.
4. M. H. Kwack, *Generalization of the big Picard theorem*, Ann. of Math. (2) **90** (1969), 9–22.
5. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
6. W. Rothstein und K. Kopfermann, *Funktionentheorie mehrerer komplexer Veränderlicher*, Bibliographisches Institut, Mannheim, 1982.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, SF-00100 HELSINKI 10, FINLAND