AN EXTENSION THEOREM FOR NORMAL FUNCTIONS

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ABSTRACT. Given a domain \( \Omega \subseteq \mathbb{C}^n \), an analytic subvariety \( V \) of \( \Omega \) and a normal function \( f: \Omega \setminus V \to \mathbb{C} \), we show that \( f \) can be extended to a holomorphic mapping \( f^*: \Omega \to \widehat{\mathbb{C}} \) provided the singularities of \( V \) are normal crossings.

1. As an extension of the big Picard theorem, Lehto and Virtanen showed [5, Theorem 9] that isolated singularities are removable for normal meromorphic functions. It is the purpose of this note to give a generalization of this result for functions defined in subdomains of \( \mathbb{C}^n \). It is conceivable that the notion of normality can be generalized in various ways to higher dimensions. Here we adopt the definition of Cima and Krantz [1, p. 305].

Let \( D \subseteq \mathbb{C} \) be the open unit disc, and let \( \Omega \subseteq \mathbb{C}^n \) be a domain. The infinitesimal form of the Kobayashi metric for \( \Omega \) at \( z \in \Omega \) in the direction \( \xi \in \mathbb{C}^n \) is defined to be

\[
F_{\Omega}(z, \xi) = \inf \left\{ \frac{|\xi|}{|f'(0)|} \mid f: D \to \Omega \text{ holomorphic}, f(0) = z, f'(0) \text{ is a positive multiple of } \xi \right\}.
\]

Here \( | \cdot | \) stands for the Euclidean length. Further, let \( z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) and let \( \eta \in \mathbb{C} \) be thought of as a tangent vector to \( \widehat{\mathbb{C}} \) at \( z \). Then the infinitesimal form of the spherical metric at \( z \) is defined by

\[
|\eta|_{\text{sph}, z} = |\eta|/(1 + |z|^2).
\]

Now suppose that \( f \) is a holomorphic mapping of \( \Omega \) into \( \widehat{\mathbb{C}} \), and let \( f' \) denote the matrix \( (\partial f/\partial z_i) \). Then \( f \) is said to be normal provided there exists a constant \( C \) such that

\[
|f'(z) \cdot \xi|_{\text{sph}, f(z)} \leq C \cdot F_{\Omega}(z, \xi) \quad \text{for all } z \in \Omega \text{ and all } \xi \in \mathbb{C}^n.
\]

The minimum of those constants \( C \), for which (1) holds true, is called the order of normality of \( f \) and denoted by \( C_f \).

REMARK 1. Suppose \( f \) is a holomorphic mapping of \( \Omega \) into \( \widehat{\mathbb{C}} \) such that \( \widehat{\mathbb{C}} \setminus f(\Omega) \) contains three points \( a_1, a_2 \) and \( a_3 \). Then \( f \) is normal. First, the distance-decreasing property of the Kobayashi metric yields

\[
F_{\widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3\}}(f(z), f'(z) \cdot \xi) \leq F_{\Omega}(z, \xi) \quad \text{for all } z \in \Omega \text{ and all } \xi \in \mathbb{C}^n.
\]
Further, making use of the homogeneity and the continuity of $F_{C\setminus\{a_1,a_2,a_3\}}$ and the spherical metric as well as the fact that

$$\left|l_{\text{sph, }z}/F_{C\setminus\{a_1,a_2,a_3\}}(z,1)\right| \to 0 \quad \text{as } z \to a_j, \ j = 1, 2, 3,$$

one readily finds a constant $C$ such that

$$|\eta|_{\text{sph, }z} \leq C \cdot F_{C\setminus\{a_1,a_2,a_3\}}(z,\eta) \quad \text{for all } z \in C\setminus\{a_1,a_2,a_3\} \text{ and all } \eta \in C.$$

Combining (2) and (3) gives the assertion. Another deduction of this result is given in [1, p. 308].

2. Set $D^* = D\setminus\{0\}$, and let $f: D^* \to C$ be normal. As noted before, $f$ extends to a function $f^*$ meromorphic in $D$ [5, p. 62]. Since $F_D(z,\eta)$ and $F_{D^*}(z,\eta)$ are comparable near $\partial D$, $f^*$ is normal in $D$. Moreover, the order of normality of $f^*$ does not deviate too much from that of $f$. More precisely, we have

**LEMMA 1.** Given a positive number $K$, there is a positive number $K'$ such that any function $f$ normal in $D^*$ with $C_f \leq K$ extends to a function $f^*$ normal in $D$ with $C_{f^*} \leq K'$.

**PROOF.** We begin with a quick proof of the Lehto-Virtanen extension theorem. Let $f$ be normal in $D^*$. Recall that

$$F_{D^*}(z,\xi) = \text{hyperbolic metric of } D^* = \frac{|\xi|}{|z| \log(1/|z|)}.$$

Hence the hyperbolic area of $D^*(r) = \{z \in C|0 < |z| < r\}$ is finite for every $r < 1$. By (1), we have

$$\int\int_{D^*(r)} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \, dx \, dy < \infty.$$

By the big Picard theorem, the singularity at 0 is inessential.

Fix $K > 0$, and let $f: D^* \to C$ be normal with $C_f \leq K$. Again, let $f^*$ stand for the extended function. Since the spherical metric is invariant under the rotations of the sphere, we may assume that $f^*(0) = 0$. Clearly, it is sufficient to exhibit an $R$, $0 < R < 1$, depending only on $K$, such that $|f^*(z)| < 1$ for $z \in D(R) = \{z \in C||z| < R\}$. By assumption,

$$\frac{|f'(z)|}{1+|f(z)|^2} \leq K \cdot \frac{1}{|z| \log(1/|z|)} \quad \text{for all } z \in D^*.$$

Set $\gamma_r = \{z \in C||z| = r\}$, $0 < r < 1$, and let $s(f^*(\gamma_r))$ denote the spherical length of $f^*(\gamma_r)$. We claim that $R = e^{-8K}$ does the job. Suppose, on the contrary, that $|f^*(z)| \geq 1$ for some $z \in D(R)$. Pick out $z_0 \in D(R)$ such that $|f^*(z_0)| = 1$ and $|f^*(z)| < 1$ for $|z| < |z_0|$. A simple estimate based on (4) gives $s(f^*(\gamma_{|z_0|})) < \pi/4$. Since the spherical distance of 0 and $f(z_0)$ is $\pi/2$, $f^*(\gamma_{|z_0|})$ lies in the half plane $\Re f(z_0) > 0$. Therefore, the winding number of $f^*(\gamma_{|z_0|})$ with respect to 0 (in C) is 0. This contradiction with $f^*(0) = 0$ completes the proof. □

The next lemma is readily deduced by elementary considerations on the Kobayashi metric. We omit the proof.
LEMMA 2. Let $f : (D^*)^n \to \hat{C}$ be normal.

1. For every $k \in \{1, \ldots, n\}$ and every $(a_1, \ldots, a_{n-1}) \in (D^*)^{n-1}$, the map $z \mapsto f(a_1, \ldots, a_{k-1}, z, a_k, \ldots, a_{n-1}), D^* \to \hat{C}$, is a normal function.

2. For every $k \in \{1, \ldots, n\}$ and every $a \in D^*$ the map $(z_1, \ldots, z_{n-1}) \mapsto f(z_1, \ldots, z_{k-1}, a, z_k, \ldots, z_{n-1}), (D^*)^{n-1} \to \hat{C}$, is a normal function.

3. For every $a = (a_1, \ldots, a_n) \in \partial D^n$ with $a_i \neq 0$, $i = 1, \ldots, n$, the map $z \mapsto f(a_1 z, \ldots, a_n z), D^* \to \hat{C}$, is a normal function.

Further, the orders of normality of all these functions are bounded above by that of $f$.

3. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $V \subset \Omega$ be an analytic subvariety of codimension one. The singularities of $V$ are said to be normal crossings provided $\Omega \setminus V$ is locally biholomorphic to $(D^*)^k \times D^{n-k}$ for some $k \in \{0, \ldots, n\}$. Our main theorem reads as follows.

THEOREM 1. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $V \subset \Omega$ be an analytic subvariety of codimension one, whose singularities are normal crossings. Suppose $f : \Omega \setminus V \to \hat{C}$ is normal. Then $f$ extends to a holomorphic mapping $f^* : \Omega \to \hat{C}$.

PROOF. Since the problem is of a local nature and the inclusion mapping is distance-decreasing (in the Kobayashi metrics), we may assume that $\Omega = D^n$ and $D^n \setminus V = (D^*)^n$.

The proof will be by induction on $n$. The case $n = 1$ is part of Lemma 1. So let $n \geq 2$ and assume the extension is possible for $1, \ldots, n-1$. Let $a = (a_1, \ldots, a_n) \in V \setminus \{0\}$, and choose $k \in \{1, \ldots, n\}$ such that $a_k \neq 0$. Consider the mapping $f_{ak} : (z_1, \ldots, z_{n-1}) \mapsto f(z_1, \ldots, z_{k-1}, a_k, z_k, \ldots, z_{n-1}), (D^*)^{n-1} \to \hat{C}$. It follows from Lemma 2 that $f_{ak}$ is normal. Hence, by the induction hypothesis, $f_{ak}$ admits a holomorphic extension $f^*_{ak} : D^{n-1} \to \hat{C}$. We set $f^*(a) = f^*_{ak}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. We will show that the extended mapping is holomorphic on $D^n \setminus \{0\}$. By the Riemann extension theorem, it suffices to prove that $f^*$ is continuous, i.e., $\text{Cl}(f; a)$, the cluster set of $f$ at any $a \in V \setminus \{0\}$ reduces to a singleton (of course, this also shows that the extension does not depend on the choice of $k$).

So let $a$, $k$ and $f^*(a)$ be as above. Pick $\varepsilon > 0$. Set

$$
\chi(z, w) = \frac{|z - w|}{(1 + |z|^2)^{1/2}(1 + |w|^2)^{1/2}} \quad \text{for } z, w \in \hat{C}
$$

and $B(a, \delta) = \{z \in \mathbb{C}^n \mid |z - a| < \delta\}$. By Lemma 2, the family

$$
\{z \mapsto f(b_1, \ldots, b_{k-1}, z, b_k, \ldots, b_{n-1}) \mid (b_1, \ldots, b_{n-1}) \in (D^*)^{n-1}\}
$$

is equicontinuous at $a_k$. Hence there exists a positive $\delta$ such that

$$
\chi(f(z), f(z_1, \ldots, z_{k-1}, a_k, z_{k+1}, \ldots, z_n)) < \varepsilon/2
$$

and

$$
\chi(f(z_1, \ldots, z_{k-1}, a_k, z_{k+1}, \ldots, z_n), f^*(a)) < \varepsilon/2 \quad \text{for } z = (z_1, \ldots, z_n) \in B(a, \delta) \setminus V.
$$

Thus $\chi(f(z), f^*(a)) < \varepsilon$ for $z \in B(a, \delta) \setminus V$. It follows that $\text{Cl}(f; a) = f^*(a)$.

It remains to extend $f$ to 0. First, we infer from Lemmas 2 and 1 that $f(z_1, \ldots, z)$ tends to a limit, say $w_0$, as $z \to 0$. It suffices to show that $\text{Cl}(f; 0) = w_0$. Let $\varepsilon > 0$. 
Consider the mappings described in Lemma 2 (3), or rather their counterparts for the restrictions of $f$ to the hyperplanes of the form $\{z_n = a\}$, $a \in D^*$. By Lemma 1, all of them extend holomorphically to 0. Moreover, it follows from Lemmas 2 and 1 that the extensions constitute an equicontinuous family at 0 ($\in \mathbb{C}$). Therefore, we find a positive $\delta$ such that $\chi(f(z), f^*(0, \ldots, 0, z_n)) < \varepsilon/3$, $\chi(f^*(0, \ldots, 0, z_n), f(z_n, \ldots, z_n)) < \varepsilon/3$ and $\chi(f(z_n, \ldots, z_n), w_0) < \varepsilon/3$ for $z = (z_1, \ldots, z_n) \in B(0, \delta) \setminus V$. Therefore $\chi(f(z), w_0) < \varepsilon$ for $z \in B(0, \delta) \setminus V$. Hence $Cl(f; 0) = w_0$. This completes the proof.

REMARK 2. Set $V = \{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 (z_1 - z_2) = 0\}$ and consider the mapping $f: D^2 \setminus V \to \mathbb{C}$, $(z_1, z_2) \mapsto z_1/z_2$. Since $f$ omits the values 0, 1 and $\infty$ in $D^2 \setminus V$, it is normal by Remark 1. Yet $f$ does not extend to a holomorphic map $D^2 \to \mathbb{C}$. Accordingly, we cannot dispense with some restrictions on the singularities of $V$ in Theorem 1.

REMARK 3. One may ask whether the extended function is normal in $\Omega$ (provided $\Omega$ is hyperbolic). However, it seems that this need not be the case even in dimension one.

REMARK 4. Results related to Theorem 1 can be found in [2, 3 and 4]. Cf. in particular [2, Theorem 2].

In the counterexample discussed above the function involved is the restriction to $D^2 \setminus V$ of a meromorphic function, i.e., a function with "indeterminacies". This is always the case as shown by

THEOREM 2. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $V$ be a subvariety of $\Omega$. Suppose $f: \Omega \setminus V \to \mathbb{C}$ is normal. Then $f$ extends to a meromorphic function in $\Omega$.

PROOF. Denote by $S(V)$ the set of singular points of $V$. By Theorem 1 $f$ extends to a holomorphic mapping of $\Omega \setminus S(V)$ into $\mathbb{C}$. Thus $f$ can be regarded as a meromorphic function in $\Omega \setminus S(V)$. Since $\dim S(V) \leq n - 2$ [6, p. 144], $f$ extends to a function meromorphic in $\Omega$ [6, p. 149].

REFERENCES


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