PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

SINGULAR INTEGRALS IN PRODUCT DOMAINS AND THE METHOD OF ROTATIONS

DONALD KRUG

(Communicated by Richard R. Goldberg)

ABSTRACT. Singular integrals with kernels of the form $K(x,y)$ where K satisfies conditions to be a bounded singular integral operator in each of its variables have been much studied lately. In this paper we use the classical method of rotations to give a proof that kernels of the form $K(x,y) = \Omega(x,y)/|x|^n|y|^m$ correspond to bounded singular integral operators.

The purpose of this paper is to use the method of rotations to give a simple proof that Calderón-Zygmund type operators when generalized to product domains are bounded operators. In particular we consider kernels of the type

$$K(x,y) = \frac{\Omega(x,y)}{|x|^n|y|^m} \tag{1}$$

for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and Ω satisfying certain conditions (which make it a C-Z kernel in each variable.) We are asking if $\|K * f\|_p \leq C_p\|f\|_p$.

If $\Omega(x,y) = \Omega_1(x)\Omega_2(y)$ where Ω_1 and Ω_2 correspond to bounded operators on L^p then we can simply iterate one variable methods. In the case above this approach does not work. Kernels $K(x,y)$ not of the form of (1) but satisfying size and smoothness conditions like those of $1/xy$ have been much studied lately (see [**2**, **3**]). The kernels we will study in this paper are less general but can be handled entirely with single variable methods.

Before proceeding I want to thank Alberto Torchinsky for suggesting this approach.

We will proceed to use the method of rotations by studying even and odd kernels.

THEOREM 1. *Let $K(x,y) = \Omega(x,y)/|x|^n|y|^m$, Ω odd in both variables, homogeneous of degree zero and $\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} |\Omega(x',y')|\,dx'\,dy' < \infty$. (Here Σ_{n-1} denotes the unit sphere in $\mathbf{R}^n$ and $x' = x/|x|$.)*

If $T_{\varepsilon,\eta}(f)(x,y) = \int_{|s|>\varepsilon}\int_{|t|>\eta} f(x-s,y-t)K(s,t)\,ds\,dt$ then

$$\left\|\sup_{\varepsilon,\eta} |T_{\varepsilon,\eta}(f)|\right\|_p \leq C_p\|f\|_p, \qquad 1 < p < \infty.$$

PROOF. Using polar coordinates, let $s = r_1 s'$, $t = r_2 t'$, then

$$\begin{aligned} K_{\varepsilon,\eta} * f(x,y) &= \int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} \Omega(s',t') \int_\varepsilon^\infty \int_\eta^\infty \frac{f(x-r_1s', y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt' \\ &= -\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} \Omega(s',t') \int_\varepsilon^\infty \int_\eta^\infty \frac{f(x+r_1s', y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt', \end{aligned}$$

Received by the editors July 10, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42B20.

since Ω is odd in the first variable. So the above expression equals

$$\frac{1}{2}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t') \times \int_{\varepsilon}^{\infty}\int_{\eta}^{\infty}\frac{f(x-r_1s',y-r_2t')-f(x+r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'$$
$$=\frac{1}{2}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t')\int_{|r_1|>\varepsilon}\int_{\eta}^{\infty}\frac{f(x-r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'.$$

Doing the same in the second variable we obtain

$$\frac{1}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t')\int_{|r_1|>\varepsilon}\int_{|r_2|>\eta}\frac{f(x-r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'.$$

Let S be the hyperplane perpendicular to s', and T to t'. Let $x=z+\lambda s'$ with $z\in S$, $y=w+\mu t'$ with $w\in T$. Then

$$=\frac{1}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t') \times\int_{|r_1|>\varepsilon}\frac{1}{r_1}\left(\int_{|r_2|>\eta}\frac{f(z+(\lambda-r)s',w+(\mu-r)t')}{r_2}\,dr_2\right)dr_1\,ds'\,dt'.$$

So

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}(f)|\right\|_p\le\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}|\Omega(s',t')|$$
$$\times\left(\int_T\int_S\int_{\mathbf{R}}\int_{\mathbf{R}}\sup_{\varepsilon>0}\left|\int_{|r_1|>\varepsilon}\frac{1}{r_1}\right.\right.$$
$$\left.\left.\times\left(\sup_{\eta>0}\left|\int_{|r_2|>\eta}\frac{f(z+(\lambda-r_1)s',w+(\mu-r_2)t')}{r_2}\right)dr_2\right|dr_1\right|^p d\lambda\,d\mu\,dw\,dz\right)^{1/p}ds'\,dt'$$
$$\le\frac{C_p}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}|\Omega(s',t')|\,ds'\,dt'\|f\|_p\le C_p\|f\|_p,$$

using the boundedness of the Maximal Hilbert transform twice. □

Exploiting this method we can in fact obtain

THEOREM 2. *Let $K(x,y)=\Omega(x',y')/|x|^n|y|^m$, where Ω is odd in the x'-variable. Let $K_x(y)=K(x,y)$ and $T_\varepsilon^x(f)(y)=\int_{|t|>\varepsilon}f(x,y-t)K_x(t)\,dt$. If*

$$(*)\qquad \|\sup|T_\varepsilon^x(f)|\|_p\le C_p\|f\|_p,$$

C_p independent of x (i.e., K is a bounded C-Z kernel in y independent of x), then

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}(f)|\right\|_p\le C_p\|f\|_p.$$

PROOF. Proceeding as above, but in the x-variable only,

$$K_{\varepsilon,\eta}*f(x,y)=\frac{1}{2}\int_{\Sigma_{n-1}}\int_{|t|>\eta}\frac{\Omega(s',t)}{|t|^m}\int_{|r|>\varepsilon}\frac{f(x-rs',y-t)}{r}\,dr\,dt\,ds'.$$

Let $x = z + \lambda s'$, $x \in S$, where S is perpendicular to s',

$$= \frac{1}{2}\int_{\Sigma_{n-1}}\int_{|t|>\eta}\frac{\Omega(s',t)}{|t|^m}\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y-t)}{r}\,dr\,dt\,ds'.$$

So

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}|\right\|_p \leq \frac{1}{2}\int_{\Sigma_{n-1}}\left(\int_S\int_{\mathbf{R}}\int_{\mathbf{R}^m}\sup_{\eta>0}\left|\int_{|t|>\eta}\frac{\Omega(s',t)}{t}\right.\right.$$
$$\left.\left.\times\sup_{\varepsilon>0}\left|\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y-t)}{r}\,dr\right|dt\right|^p dy\,d\lambda\,dz\right)^{1/p} ds'.$$

Now using the assumption (*), that our operator is bounded as an operator acting only in the second variable, we have

$$\leq C_p\int_{\Sigma_{n-1}}\left(\int_{\mathbf{R}^m}\int_S\int_{\mathbf{R}}\sup_{\varepsilon>0}\left|\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y)}{r}\,dr\right|^p d\lambda\,dz\,dy\right)^{1/p} ds'$$
$$\leq C_p\int_{\Sigma_{n-1}}\left(\int_{\mathbf{R}^m}\int_S\int_{\mathbf{R}}|f(x,y)|^p\,dx\,dy\right)^{1/p} ds' = C_p\left(\int_{\Sigma_{n-1}}ds'\right)\|f\|_p. \quad \square$$

THEOREM 3. *Again let $K(x,y) = \Omega(x,y)/|x|^n|y|^m$, Ω homogeneous of degree zero in each variable, $\int_{\Sigma_{n-1}}\Omega(s,t)\,ds = 0$ a.e. in t, $\int_{\Sigma_{m-1}}\Omega(s,t)\,dt = 0$ a.e. in s, $(\int_{\Sigma_{n-1}}|\Omega(s,t)|^2\,ds)^{1/2} < C$ independent of t, and $(\int_{\Sigma_{m-1}}|\Omega(s,t)|^2\,dt)^{1/2} < C$ independent of s. Then $\|K*f\|_p \leq C_p\|f\|_p$.*

PROOF. We may assume Ω is even in both variables (since the hypotheses assure that both the previous theorems hold) and that $f = \rho$, a testing function. Let y be fixed, $K_y(x) = \Omega(x',y')/|x|^n$, $T_y(\rho) = K_y * \rho$ and R_i be the ith Riesz Transformation in x. Then [**4**, p. 225] shows that R_iT_y is essentially an odd C-Z operator. In fact, it is shown that $(R_iT_y)^\wedge = (J_i^y)^\wedge$ where $J_i^y(x) = \omega_{y'}(x')/|x|^n$, $\omega \in L^2(\Sigma_{n-1},dx)$ and $\omega_y(x)$ is odd in x.

If we set

$$J_i(x,y) = \frac{J_i^y(x)}{|y|^m} = \frac{\omega(x',y')}{|x|^n|y|^m}, \quad \text{where } \omega(x',y') = \omega_{y'}(x'),$$

then J_i is an odd C-Z integral operator in the x-variable.

In the y-variable we see that J_i is a C-Z operator (as expected) since

$$[(R_jT)\rho]^\wedge = -i\frac{x_j}{|x|}\Omega_0\left(\frac{x}{|x|},\frac{y}{|y|}\right)\hat{\rho}$$

where $\Omega_0 = \widehat{K}$ (see [**4**]). In the y-variable this acts exactly as did $K*\rho$ (up to a constant C_x, $|C_x| < 1$) and so is still a C-Z operator.

So by Theorem 2 above, $\|J_i * \rho\|_p \leq C\|\rho\|_p$.

It follows that

$$\|K*\rho\|_p = \|\Sigma R_i * J_i * \rho\|_p \leq \sum\|R_i * (J_i*\rho)\|_p \leq C\|\rho\|_p. \quad \square$$

There is a weighted version of this as well.

For one variable $x \in \mathbf{R}^n$, a function $w(x) > 0$ is called an A_p-weight, $1 < p < \infty$, if it satisfies

$$\frac{1}{|Q|}\int_Q w(x)\,dx\left(\frac{1}{|Q|}\int_Q [w(x)]^{-1/(p-1)}\,dx\right)^{p-1} \le B,$$

for some $B < \infty$ and all cubes Q in $\mathbf{R}^n$.

The following theorem is well known.

THEOREM. *If w is an A_p-weight and K is a standard C-Z kernel then*

$$\int_{\mathbf{R}^n} |f * K(y)|^p w(y)\,dy \le C(p,B)\int_{\mathbf{R}^n} |f(y)|^p w(y)\,dy.$$

See [**1**] *for results on weights.*

In the work above we used only that $T = K * f$ was bounded in L^p-norm in each variable separately. Thus if $w(x,y)$ is an A_p-weight in each variable, i.e., if $w(x,y) > 0$, $x \to w(x,y)$ is an A_p-weight with B independent of y and similarly for $y \to w(x,y)$ then following the method of proof above we have the following.

THEOREM 4. *If w is as above and K satisfies the conditions of Theorem* 3 *then*

$$\left(\int_{\mathbf{R}^n\times\mathbf{R}^m} |K * f(y)|^p w(y)\,dy\right)^{1/p} \le C(p,A)\left(\int_{\mathbf{R}^n\times\mathbf{R}^m} |f(y)|^p w(y)\,dy\right)^{1/p}.$$

REFERENCES

1. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
2. R. Fefferman, *Singular integrals on product H^p-spaces*, preprint.
3. R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math. **45** (1982), 117–143.
4. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN KENTUCKY UNIVERSITY, HIGHLAND HEIGHTS, KENTUCKY 41076