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THE DIFFERENTIABILITY OF THE HAIRS OF $\exp(Z)$

M. VIANA DA SILVA

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ABSTRACT. We prove that the hairs of the exponential-like maps $f(z) = \lambda e^z$ are smooth curves. This answers affirmatively a question of Devaney and Krych. The proof is constructive in the sense that a dynamically defined C^∞ parametrization is presented.

0. Introduction. The study of the dynamical behaviour of the complex exponential map was begun by Fatou in 1926, following the work of Julia and Fatou himself concerning the dynamics of the rational maps on the sphere.

Recently Devaney and Krych [**DK**] were able to give a symbolic description of that behaviour, by introducing the idea of "itinerary of a complex number under the action of exp" (see §1 or [**DK**] for the definition). This led them to define "hair of exp associated to a given itinerary" as, roughly, the set of complex numbers sharing that itinerary and having fastly growing iterates. The hairs turn out to be quite simple sets (in particular they are curves) and have very simple dynamical properties.

In fact these properties may be used to define "hair of an entire transcendental function" and it turns out that hairs really exist for a very large class of such functions (see [**DT**]). This, naturally, makes them an important tool for understanding the dynamics of these functions and increases the interest in their study.

In this note we complement the description of the hairs of exp given in [**DK**], proving that they are C^∞-differentiable curves. To do this we construct in §2 a parametrization β of a hair as the uniform limit of a sequence $(\beta_{n,0})_n$ and, in §3, we prove β to be C^∞ by showing that all the sequences of derivatives $(\beta_{n,0}^{(k)})_n$ are also uniformly convergent.

As far as we know it is still an open question whether the hairs are or are not analytic. We do not even know if β is an analytical parametrization.

I would like to thank A. Douady for suggesting me this problem and for helpful conversations on the subject.

1. Definitions and results. Let $\lambda \in \mathbf{C} - \{0\}$ and take $\lambda = \rho e^{i\theta}$, with $\rho > 0$ and $-\pi \leq \theta < \pi$.

Let $f(z) = \lambda e^z$, $z \in \mathbf{C}$ and $g(x) = \rho e^x$, $x \in \mathbf{R}$. Take, for $s \in \mathbf{Z}$, $B(s) = \{z : (2s-1)\pi < \operatorname{Im} z + \theta < (2s+1)\pi\}$. Since $f|_{B(s)}$ is a diffeomorphism, denote $f_s^{-1} = (f|_{B(s)})^{-1}$.

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DEFINITION. The f-itinerary of $z \in \mathbf{C}$ is the integer sequence $S(z) = (s_n)_n$ defined by

$$2(s_n - 1)\pi \le \operatorname{Im} f^n(z) + \theta < 2(s_n + 1)\pi, \qquad n \ge 0.$$

REMARK 1. If $S(z) = (s_n)_n$ then $S(f(z)) = (s_{n+1})_n$. This means that $S\colon C \to \mathbf{Z}^{\mathbf{N}}$ conjugates f to the shift map $\sigma((s_n)_n) = (s_{n+1})_n$.

REMARK 2. If $(s_n)_n$ is the f-itinerary of some $z \in \mathbf{C}$ then there is $\hat{x} \in \mathbf{R}$ such that $(*)$ $(2|s_n| + 1)\pi + |\theta| \le g^n(\hat{x})$, for all $n \ge 0$. This is proved in the same way as Proposition 1.2 in [**DK**], with $\hat{x} = |z| + 2\pi$. It follows from our results that $(*)$ is also a sufficient condition for an integer sequence to be a f-itinerary (see also [**DK**]).

DEFINITION. Let $\mathbf{s} = (s_n)_n$ and $\hat{x} \in \mathbf{R}$ satisfy condition $(*)$.

(1) The *tail of hair* of f associated to $\mathbf{s}$ is defined by $T = T(\mathbf{s}) = \{z : S(z) = \mathbf{s}$ and, for all $n \ge 0$, $\operatorname{Re} f^n(z) \ge g^n(\hat{x})\}$. Let $\mathbf{s}^k = (s_{n+k})_n$ represent the kth shift of $\mathbf{s}$. By Remark 1 $f^k(T) \subset T(\mathbf{s}^k)$, so the following makes sense.

(2) The *hair* of f associated to $\mathbf{s}$ is defined by

$$C = C(\mathbf{s}) = \bigcup_{k \ge 0} C_k(\mathbf{s}),$$

where $C_k(\mathbf{s})$ is the connected component of $f^{-k}(T(\mathbf{s}^k))$ containing T.

Here we prove

THEOREM A. *There is a homeomorphism $\beta = \beta(\mathbf{s}) : [\hat{x}, +\infty[\to T$ which is C^∞-differentiable on $]\hat{x}, +\infty[$ and such that $0 < |\beta'(x)| \le 1$, for all $x > \hat{x}$. Moreover, for all itinerary $\mathbf{s}$, we have the relation $f \circ \beta(\mathbf{s}) = \beta(\mathbf{s}^1) \circ g$.*

From this easily follows:

THEOREM B. *C is a differentiable curve.*

PROOF. Just note that, by taking appropriate backward images, $f^{-k} \circ \beta(\mathbf{s}^k)$ is a C^∞ parametrization of $C_k(\mathbf{s})$, whose restriction to $[g^k(\hat{x}), +\infty[$ is $\beta \circ g^{-k}$.

2. Construction of β.

DEFINITION For $0 \le p \le n$ $\beta_{n,p}$ is the (smooth) curve defined for $x \ge \hat{x}$, by

$$\beta_{n,n}(x) = g^n(x) + (2\pi s_n - \theta)i; \quad \beta_{n,p}(x) = f_{s_p}^{-1}(\beta_{n,p+1}(x)), \qquad 0 \le p < n.$$

The following lemma is easily proved by induction on p, starting with $p = n$ and counting down. Incidentally, it justifies the above claim that the $\beta_{n,p}$ are well-defined smooth curves.

LEMMA 2.1. *For all $0 \le p \le n$ and $x \ge \hat{x}$*
(i) $\operatorname{Re} \beta_{n,p}(x) \ge g^p(x)$,
(ii) $|\operatorname{Im} \beta_{n,p}(x) - (2\pi s_p - \theta)| < \pi/2$.

LEMMA 2.2. *Let $z_1, z_2 \in \mathbf{C}$ be such that $|\operatorname{Im} z_1 - \operatorname{Im} z_2| < \pi$. Then $|f(z_1) - f(z_2)| \ge a|z_1 - z_2|$, with $a = \min\{\operatorname{Re} f(z_1), \operatorname{Re} f(z_2)\}$.*

PROOF. Let c_1 be the straight-line segment from $f(z_1)$ to $f(z_2)$ and c_0 be the smooth curve from z_1 to z_2 such that $f(c_0) = c_1$. It is easy to check that

$$|f(z_1) - f(z_2)| = \int_0^1 |c_1'(t)|\,dt = \int_0^1 |c_1(t)| \cdot |c_0'(t)|\,dt$$
$$\geq a \int_0^1 |c_0'(t)|\,dt \geq a|z_1 - z_2|. \quad \square$$

PROPOSITION 2.3. *For all* $0 \leq p < n$ *and* $x \geq \hat{x}$

$$|\beta_{n,p}(x) - \beta_{n-1,p}(x)| \leq g^n(\hat{x})/[g^{p+1}(x) \cdots g^n(x)].$$

PROOF. Note that

$$g^n(\hat{x}) \geq |(g^n(x) + (2\pi s_n - \theta)i) - g^n(x)| = |f^{n-p}(\beta_{n,p}(x)) - f^{n-p}(\beta_{n-1,p}(x))|.$$

The Proposition now follows by using $(n-p)$ times the preceding lemmas.

It follows that, for all $p \geq 0$, the sequence $(\beta_{n,p})_n$ is uniformly convergent on $[\hat{x}, +\infty[$. We define

DEFINITION. $\beta = \lim_n \beta_{n,0}$.

Trivially β is continuous and, from Lemma 2.1, $\beta([\hat{x}, +\infty[) \subset T$. On the other hand, if we take for $z \in T$,

$$\eta(z) = \sup\{x \geq \hat{x} : \text{ for all } n \geq 0 \ \operatorname{Re} f^n(z) \geq g^n(x)\}$$

then

(A) $\eta \circ \beta = \mathrm{id}$,

(B) $\beta \circ \eta = \mathrm{id}$, and this clearly implies that β is a homeomorphism.

We just give a sketch of the proof of (A) and (B). The details are easy to complete and are left to the reader.

(A) For $x \geq \hat{x}$ there is $a > 0$ such that, for all $p \geq 0$,

$$g^p(x) \leq \operatorname{Re} f^p \beta(x) \leq g^p(x) + a.$$

Then, easily, $x = \eta\beta(x)$.

(B) For $z \in T$ there is $a > 0$ such that, for all $p \geq 0$

$$g^p \eta(z) \leq \operatorname{Re} f^p(z) \leq g^p \eta(z) + a.$$

It follows, using (a) that the sequence $(f^p(z) - f^p \beta\eta(z))_p$ is bounded. Using Lemma 2.3, we get $z = \beta\eta(z)$.

3. Smoothness of β. It is easy to check that, if u is a smooth curve in $\mathbf{C}$, then for $k \geq 1$ we have

$$(\exp \circ u)^{(k)} = (\exp \circ u) \sum_{(k)} u^{(k_1)} \cdots u^{(k_i)}$$

where the last factor represents a (particular) sum of products $u^{(k_1)} \cdots u^{(k_r)}$ of derivatives of u, which is homogeneous in the sense that $k_1 + \cdots + k_r = k$, for all such products.

We begin by proving a technical lemma that will be needed later. This proof is interesting in itself because the same kind of argument will be used to prove the main results in this section:

LEMMA 3.1. *For all $k \geq 1$, there is $M_k > 0$ such that, for all $p \geq 0$ and $x \geq \hat{x}$*

$$(g^p)^{(k)}(x) \leq M_k \cdot p^{k-1}[g^p(x)]^{pk}.$$

PROOF. *First step*: It is easy to check that $(g^p)' \leq [g^p]^p$.

Inductive step: Let $\gamma_p = \sum_{j=0}^{p-1} g^j$ so that $(g^p)' = \rho^p \exp(\gamma_p)$. Then we have

$$(\gamma_p)^{(k)} = \sum_{j=0}^{p-1} (g^j)^{(k)} \leq \sum_{j=0}^{p-1} M_k j^{k-1} [g^j]^{jk} \leq M_k \cdot p^k [g^p]^{pk}$$

$$\begin{aligned}(g^p)^{(k+1)} &= \rho^p (\exp \circ \gamma_p) \sum_{(k)} \gamma_p^{(k_1)} \cdots \gamma_p^{(k_r)} \\ &\leq [g^p]^p \sum_{(k)} M_{k_1} p^{k_1} [g^p]^{pk_1} \cdots M_{k_r} p^{k_r} [g^p]^{pk_r} \\ &\leq \left(\sum_{(k)} M_{k_1} \cdots M_{k_r} \right) \cdot p^k \cdot [g^p]^{p(k+1)}. \quad \square\end{aligned}$$

Now, from $f^{n-p} \circ \beta_{n,p} = g^n + (2\pi s_n - \theta)i$ we easily get the following expression for the derivative of $\beta_{n,p}$:

$$\beta'_{n,p} = (g^p)' \cdot [g^{p+1} \cdots g^n] / [f\beta_{n,p} \cdots f^{n-p}\beta_{n,p}],$$

$$\beta'_{n,p} = (g^p)' \cdot \exp\left(\sum_{j=p}^{n-1} (g^j + (2\pi s_j - \theta)i - \beta_{n,j}) \right).$$

Let

$$\alpha_{n,p} = \sum_{j=p}^{n-1} (g^j + (2\pi s_j - \theta)i - \beta_{n,j}) \tag{1A}$$

so that

$$\beta'_{n,p} = (g^p)' \cdot \exp(\alpha_{n,p}). \tag{1B}$$

Then

$$\alpha_{n,p}^{(k)} = \sum_{j=p}^{n-1} ((g^j)^{(k)} - \beta_{n,j}^{(k)}) \tag{2A}$$

and

$$\beta_{n,p}^{(k+1)} = \sum_{l=0}^{k} \left(\binom{k}{l} (g^p)^{(k-l+1)} (\exp \circ \alpha_{n,p}) \sum_{(l)} \alpha_{n,p}^{(l_1)} \cdots \alpha_{n,p}^{(l_r)} \right). \tag{2B}$$

The following is an easy consequence of Lemma 2.1 and Proposition 2.3.

LEMMA 3.2. *For all $0 \leq p \leq n$ and $x \geq \hat{x}$*

(1) $\operatorname{Re} \alpha_{n,p}(x) \leq 0$ (*so* $|\exp(\alpha_{n,p}(x))| \leq 1$),

(2) $|\exp(\alpha_{n,p}(s)) - \exp(\alpha_{n-1,p}(x))| \leq |\alpha_{n,p}(x) - \alpha_{n-1,p}(x)| \leq n \cdot g^n(\hat{x})/g^n(x)$.

LEMMA 3.3. *For all $k \geq 1$, there are polynomials $\mathscr{P}_k, \widetilde{\mathscr{P}_k}$ such that, for $x \geq \hat{x}$,*
(i) $|\beta_{n,p}^{(k)}(x)| \leq \mathscr{P}_k(n)[g^{n-1}(x)]^{nk}$, *if* $0 \leq p < n$,
(ii) $|\alpha_{n,p}^{(k)}(x)| \leq \widetilde{\mathscr{P}_k}(n)[g^{n-1}(x)]^{nk}$, *if* $0 \leq p \leq n$.

PROOF. *First step*: $|\beta'_{n,p}| = |(g^p)' \exp(\alpha_{n,p})| \leq [g^p]^p \leq [g^{n-1}]^n$.
Inductive step:

$$|\alpha_{n,p}^{(k)}| \leq \sum_{j=p}^{n-1}((g^j)^{(k)} + |\beta_{n,j}^{(k)}|) \leq \sum_{j=p}^{n-1}(M_k j^{k-1}[g^j]^{jk} + \mathscr{P}_k(n)[g^{n-1}]^{nk})$$

$$\leq n(M_k n^{k-1} + \mathscr{P}_k(n))[g^{n-1}]^{nk},$$

$$|\beta_{n,p}^{(k+1)}| \leq \sum_{l=0}^{k}\binom{k}{l}M_{k-l+1}p^{k-l}[g^p]^{p(k-l+1)}\sum_{(l)}\widetilde{\mathscr{P}}_{k_1}(n)[g^{n-1}]^{nl_1}\cdots\widetilde{\mathscr{P}}_{k_r}(n)[g^{n-1}]^{nl_r}$$

$$\leq \left(\sum_{l=0}^{k}\binom{k}{l}M_{k-l+1}n^{k-l}\sum_{(l)}\widetilde{\mathscr{P}}_{k_1}(n)\cdots\widetilde{\mathscr{P}}_{k_r}(x)\right)[g^{n-1}]^{n(k+1)}. \quad \square$$

We are now in position to prove the main result in this section, which implies the convergence of all the sequences $(\beta_{n,p}^{(k)})_n$, and so, in particular, the C^∞-differentiability of β.

PROPOSITION 3.4. *For all $k \geq 1$ there are polynomials Q_k, $\widetilde{Q}_k$ such that, for $0 \leq p < n$ and $x \geq \hat{x}$,*
(i) $|\beta_{n,p}^{(k)}(x) - \beta_{n-1,p}^{(k)}(x)| \leq Q_k(n)[g^{n-1}(x)]^{nk}g^n(\hat{x})/g^n(x)$,
(ii) $|\alpha_{n,p}^{(k)}(x) - \alpha_{n-1,p}^{(k)}(x)| \leq \widetilde{Q}_k(n)[g^{n-1}(x)]^{nk}g^n(\hat{x})/g^n(x)$.

PROOF. *First step*:

$$|\beta'_{n,p} - \beta'_{n-1,p}| \leq (g^p)'|\exp(\alpha_{n,p}) - \exp(\alpha_{n-1,p})| \leq [g^{n-1}]^n n[g^n(\hat{x})/g^n].$$

Inductive step:

$$|\alpha_{n,p}^{(k)} - \alpha_{n-1,p}^{(k)}| \leq \sum_{j=p}^{n-1}|\beta_{n,j}^{(k)} - \beta_{n-1,j}^{(k)}| \leq n \cdot Q_k(n)[g^{n-1}]^{nk}g^n(\hat{x})/g^n.$$

The calculations for $|\beta_{n,p}^{(k+1)} - \beta_{n-1,p}^{(k+1)}|$ are long and tedious but present no difficulty at all. They are not detailed here. $\square$

The convergence of $(\beta_{n,p}^{(k)})_n$ now follows from the lemma below which can be proved by elementary means (e.g. the root test for convergence).

LEMMA 3.5. *Let Q be a polynomial. Then the series*

$$\sum_n Q(n)[g^{n-1}(x)]^{nk}g^n(\hat{x})/g^n(x)$$

is uniformly convergent on $[a, +\infty[$, for all $a > \hat{x}$.

Finally from formula (1B) above we get $\beta'_{n,0} = \exp(\alpha_{n,0})$ so, by Lemma 3.2(1), $|\beta'| = \lim|\beta'_{n,0}| \leq 1$. On the other hand, from Lemma 3.2(2) it easily follows that $(\alpha_{n,0})_n$ converges and this clearly implies $\beta' \neq 0$. This ends the proof of Theorem A.

REFERENCES

[**M**] M. Misiurewicz, *On iterates of* e^z, Ergodic Theory Dynamical Systems **1** (1981), 103–106.

[**DK**] R. L. Devaney and M. Krych, *Dynamics of* exp, Ergodic Theory Dynamical Systems **4** (1984), 35–52.

[**D1**] R. L. Devaney, *Exploding Julia sets*, Chaotic Dynamics and Fractals, (M. Barnsley and S. Demko, eds.), Notes and Reports in Mathematics in Science and Engineering, vol. 2, Academic Press, 1986.

[**D2**] ——, *Julia sets and bifurcation diagrams for exponential maps*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), 167–171.

[**DT**] R. L. Devaney and F. Tangerman, *Dynamics of entire functions near the essential singularity*, preprint, Boston University.

DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS DO PORTO, 4000-PORTO-PORTUGAL

Current address: IMPA, Estr. Dona Castorina 110, 22460, Rio de Janeiro, Brasil