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INTERSECTION OF LEBESGUE SPACES L_1 AND L_2

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ABSTRACT. The linear topological properties of the intersection of Lebesgue spaces L_1 and L_2 are investigated.

1. Introduction. Let $L_p(0,\infty)$ be the usual Lebesgue space of complex-valued functions $f(t)$ equipped with the norm

$$||f||_p = \left(\int_0^\infty |f(t)|^p\,dt\right)^{1/p} \qquad (1 \le p < \infty)$$

and $||f||_\infty = \operatorname{ess\,sup}_{t>0}|f(t)|$. The intersection of Lebesgue spaces $L_1(0,\infty) \cap L_2(0,\infty)$ (denoted $L_1 \cap L_2$ for the sake of brevity) is a Banach space when given the norm $||f||_{L_1\cap L_2} = \max(||f||_1, ||f||_2)$. This class of functions arises, of course, in elementary harmonic analysis: for example, in the Plancherel Theorem and in the Marcinkiewicz Interpolation Theorem (see e.g. [**13**]). In this note we study the properties of $L_1 \cap L_2$ as a Banach space, particularly in connection with a theorem of Pełczyński on the impossibility of embedding $L_1(0,1)$ into a Banach space with unconditional basis [**12**]. These results shed some light on the linear topological properties of the Hardy space of analytic functions, $H_1(D)$.

The idea of studying $L_1 \cap L_2$ originated in the paper [**2**] of mine and N. L. Carothers (see also [**3**, **4**, **5**, **8**]), in which intersections of Lebesgue spaces are used to prove moment inequalities in Lorentz $L_{p,q}$ spaces. I should like to record my gratitude to N. L. Carothers for helpful discussion and for stimulating my interest in these questions.

Notation and terminology are standard and agree with, for example, those of [**6**].

2. The Banach space $L_1 \cap L_2$.

PROPOSITION 1. *Let X be a subspace of $L_1 \cap L_2$. Then X is isomorphic to a Hilbert space and complemented in $L_1 \cap L_2$, or X contains a subspace isomorphic to l_1 and complemented in $L_1 \cap L_2$.*

PROOF. First suppose that the $L_1 \cap L_2$ and the $L_2(0,\infty)$ topologies agree on X. Then X is isomorphic to a Hilbert space and the orthogonal projection from $L_2(0,\infty)$ onto X restricts to a bounded projection from $L_1 \cap L_2$ onto X. If the topologies do not agree, then given $\varepsilon > 0$ there exists $f \in X$ such that $||f||_1 = 1$ and

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$||f||_2 < \varepsilon$. By Hölder's inequality $||f|_{[0,M]}||_1 \leq \sqrt{M}||f||_2 \leq \sqrt{M}\varepsilon$, for all $M > 0$. So by an inductive procedure one can construct functions $(f_n)_{n=1}^\infty$ in X and disjoint compactly supported functions $(g_n)_{n=1}^\infty$ in $L_1 \cap L_2$ such that $||g_n||_1 = 1$, $||g_n||_2 \leq \varepsilon_n$, and $||f_n - g_n||_{L_1 \cap L_2} < \varepsilon_n$, where $(\varepsilon_n)_{n=1}^\infty$ is any null sequence of positive numbers. Then $(g_n)_{n=1}^\infty$ is equivalent in $L_1 \cap L_2$ to the unit vector basis of l_1, and its closed linear span is the range of a contractive projection on $L_1(0,\infty)$ whose restriction to $L_1 \cap L_2$ is a bounded projection. A standard perturbation argument now shows that $(f_n)_{n=1}^\infty$ spans a complemented subspace of $L_1 \cap L_2$ isomorphic to l_1 provided ε_k decreases rapidly to zero.

PROPOSITION 2. (a) *$L_1 \cap L_2$ is isomorphic to a dual space and has nonseparable dual.*

(b) *$L_1 \cap L_2$ is isomorphic to a subspace of $L_1(0,1)$.*

PROOF. (a) The dual of $L_1 \cap L_2$ is $L_2(0,\infty) + L_\infty(0,\infty)$ and a predual is the closure of the simple integrable functions in $L_2(0,\infty) + L_\infty(0,\infty)$.

(b) The diagonal mapping $f \mapsto (f,f)$ defines an embedding of $L_1 \cap L_2$ into $L_1(0,\infty) \oplus L_2(0,\infty)$, and the latter space embeds into $L_1(0,1)$.

COROLLARY 3. *$L_1 \cap L_2$ has the Radon-Nikodym Property but is not a UMD space.*

PROOF. Every separable dual space has the Radon-Nikodym Property [**7**]. A UMD space cannot contain l_1 [**1**].

Let $(h_n)_{n=1}^\infty$ be the Haar system on $[0,1]$ defined by $h_1(t) \equiv 1$ and

$$h_{2^k+l}(t) = \begin{cases} 1 & \text{if } t \in [(2l-2)2^{-k-1}, (2l-1)2^{-k-1}], \\ -1 & \text{if } t \in [(2l-1)2^{-k-1}, 2l \cdot 2^{-k-1}], \\ 0 & \text{otherwise,} \end{cases}$$

for $k \geq 0$ and $1 \leq l \leq 2^k$. For $k \geq 1$, let $(h_n^k)_{n=1}^\infty$ be the Haar system translated to the interval $[k-1,k]$ and let $H = (\tilde{h}_n)_{n=1}^\infty$ be the diagonal ordering of $(h_n^k)_{k=1}^\infty{}_{n=1}^\infty$. Then H is a Schauder basis for $L_1 \cap L_2$. Now let Z be the subspace of $L_1 \cap L_2$ consisting of all functions $f(t)$ such that $\int_{n-1}^n f(t)\,dt = 0$ for all $n \geq 1$. Then the subsequence G obtained from H by deleting the terms $h_1^1, h_1^2, h_1^3, \ldots$ forms a Schauder basis for Z.

PROPOSITION 4. *Suppose that $L_1 \cap L_2$ is isomorphic to a subspace of a Banach space X having a Schauder basis $(x_n)_{n=1}^\infty$. Then G is equivalent to a block basis of $(x_n)_{n=1}^\infty$.*

PROOF. This fact is well known for the Haar system. Obvious changes to the proof given in [**10**], which uses a convexity theorem of Liapounoff, give the result.

PROPOSITION 5. *H is not an unconditional basis for $L_1 \cap L_2$.*

PROOF. Let $f_n^k = h_1^k + h_2^k + 2h_3^k + 4h_5^k + \cdots + 2^{2n+1}h_{2^{2n+1}+1}^k$ and let $g_n^k = h_1^k + 2h_3^k + 8h_9^k + 32h_{33}^k + \cdots + 2^{2n+1}h_{2^{2n+1}+1}^k$. It is easily checked that $||f_n^k||_1 = 1$, $||f_n^k||_2 = 2^{n+1}$, and that $(n+2)/4 \leq ||g_n^k||_1 \leq n+2$. For each $k \geq 3$, select n_k so that $2^{n_k} \leq \sqrt{k} \leq 2^{n_k+1}$. Consider

$$f = \sum_{k=3}^\infty \frac{f_{n_k}^k}{k(\ln k)^2} \quad \text{and} \quad g = \sum_{k=3}^\infty \frac{g_{n_k}^k}{k(\ln k)^2}.$$

Then

$$||f||_1 = \sum_{k=3}^{\infty} \frac{||f_{n_k}^k||_1}{k(\ln k)^2} = \sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2} < \infty,$$

and

$$\begin{aligned}||f||_2^2 &= \sum_{k=3}^{\infty} \frac{1}{k^2(\ln k)^4}||f_{n_k}^k||_2^2 = \sum_{k=3}^{\infty} \frac{1}{k^2(\ln k)^4}(2^{n_k+1})^2 \\ &\leq \sum_{k=3}^{\infty} \frac{4}{k^2(\ln k)^4}k = \sum_{k=3}^{\infty} \frac{4}{k(\ln k)^4} < \infty.\end{aligned}$$

So $f \in L_1 \cap L_2$. But

$$\begin{aligned}||g||_1 &= \sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}||g_{n_k}^k||_1 \geq \sum_{k=3}^{\infty} \frac{1}{4k(\ln k)^2}(n_k + 2) \\ &\geq \sum_{k=3}^{\infty} \frac{1}{8k(\ln k)^2}\frac{\ln k}{\ln 2}\end{aligned}$$

which diverges. Since g is obtained from f by multiplying the basis expansion of f with respect to H by a sequence of zeros and ones, it follows that H is not an unconditional basis.

THEOREM 6. *$L_1 \cap L_2$ does not embed isomorphically into any Banach space with unconditional basis.*

PROOF. It follows at once from Proposition 5 that G is not an unconditional basis for Z. Since a block basis of an unconditional basis is unconditional, the result now follows from Proposition 4.

Let D denote the unit disc in the complex plane, and let $H_1(D)$ be the Hardy space of analytic functions whose radial limits belong to $L_1(\partial D)$. In [**11**] Maurey observed without proof that every subspace of $L_1(\partial D)$ with unconditional basis embeds isomorphically into $H_1(D)$. Since this fact is not widely known but is plainly relevant to this paper a short proof is sketched below as Remark 9. First note the following consequence of Proposition 1 and Theorem 6.

COROLLARY 7. (a) *$L_1 \cap L_2$ does not embed isomorphically into $H_1(D)$.*
(b) *$H_1(D)$ does not embed isomorphically into $L_1 \cap L_2$.*

PROOF. (a) This follows from Theorem 6 since $H_1(D)$ has an unconditional basis [**11**].

(b) This follows from Proposition 1 since l_p embeds into $H_1(D)$ for all $1 < p < 2$ ([**9**], and Remark 9 below).

REMARK 8. Let $1 < p_n \leq 2$ and let $p_n \to 1$ as $n \to \infty$. Then $(\sum_{n=1}^{\infty} \bigoplus L_{p_n})_1$ is isometric to a dual subspace of $L_1(0,1)$ and like $L_1 \cap L_2$ it does not embed isomorphically into any space with unconditional basis [**10**]. In contrast to $L_1 \cap L_2$ this space contains every reflexive subspace of $L_1(0,1)$.

REMARK 9. Let $(f_n)_{n=1}^{\infty}$ be a normalized unconditional basic sequence in $L_1(\partial D)$. By the Stone-Weierstrass Theorem there exist analytic polynomials $(p_n)_{n=1}^{\infty}$ such that $|| \, |f_n| - |p_n| \, ||_1 < 2^{-n}$. By the Riemann-Lebesgue Lemma $p_n z^l \xrightarrow{w} 0$ as $l \to \infty$, and so we may further assume that $(p_n)_{n=1}^{\infty}$ is equivalent to a block basis of an unconditional basis of $H_1(D)$, and hence that $(p_n)_{n=1}^\infty$ is unconditional. Now Khintchine's inequality gives for all scalars $a_1, a_2, \ldots$

$$\left\|\sum_{n=1}^\infty a_n p_n\right\|_1 \sim \left\|\left(\sum_{n=1}^\infty |a_n|^2|p_n|^2\right)^{1/2}\right\|_1$$
$$\sim \left\|\left(\sum_{n=1}^\infty |a_n|^2|f_n|^2\right)^{1/2}\right\|_1 \sim \left\|\sum_{n=1}^\infty a_n f_n\right\|_1,$$

and so $(f_n)_{n=1}^\infty$ and $(p_n)_{n=1}^\infty$ are equivalent basic sequences.

REMARK 10. Let $1 < p < \infty$. It is easy to adapt the proof of Proposition 5 to the case of $L_1 \cap L_p$, and from this it follows that $L_1 \cap L_p$ does not embed into any space with an unconditional basis. It is easy to extend the other results too. For example, Proposition 1 admits the following generalization: every subspace of $L_1 \cap L_p$ embeds into $L_p(0,1)$ or contains a complemented subspace isomorphic to l_1.

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