THE CONVOLUTION OF RADON MEASURES

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Abstract. The convolution of a pair of bounded Radon measures is extended to Čech-complete topological semigroups with separately continuous multiplication.

Let $S$ be a separately continuous topological semigroup, $CB(S)$ the space of all bounded real-valued continuous functions on $S$ and $M(S)$ the set of all bounded real-valued Radon measures on $S$. For $\mu$ in $M(S)$ and $f$ in $CB(S)$, a result of Glicksberg [3] implies that the map $t \rightarrow \int f(st) \, d\mu(s)$ is continuous when restricted to the compacta of $S$. Hence it is $\nu$-measurable for any $\nu$ in $M(S)$. Let $\psi(f) = \int \int f(st) \, d\mu(s) \, d\nu(t)$ for $f$ in $CB(S)$. In this note it is shown that for a large class of semigroups, which includes the class of Čech-complete semigroups, $\psi$ is represented by a unique bounded Radon measure. Denoting this measure by $\mu \ast \nu$, we have

$$\int f \, d\mu \ast \nu = \int \int f(st) \, d\mu(s) \, d\nu(t) = \int \int f(st) \, d\nu(t) \, d\mu(s)$$

for each $f$ in $CB(S)$. This extends the convolution of a pair of bounded measures to a wide class of topological semigroups with separately continuous multiplication, including those considered in [1, 3, 6, 7].

All topological spaces are assumed to be completely regular Hausdorff. $\beta X$ denotes the Stone-Čech compactification of the topological space $X$. $f'$ is the unique extension of $f$ in $CB(X)$ to $C(\beta X)$. For the set $A \subseteq X$, $\chi_A$ is the characteristic function of $A$. $\| \|$ denotes the uniform norm.

Lemma 1. Let $S$ be a completely regular separately continuous topological semigroup. Let $F, K$ be a pair of compacta of $S$, such that $KF$ is measurable in $\beta S$ with respect to every measure in $M(\beta S)$. Then for $\mu, \nu \in M(S)$ the functional, $\psi(f) = \int_F \int_K f(st) \, d\mu(s) \, d\nu(t)$ for $f$ in $CB(S)$, is represented by a unique bounded Radon measure on $S$.

Proof. It is sufficient to prove the lemma for $\mu \geq 0$ and $\nu \geq 0$. Let $\gamma'$ be a regular finite Borel measure on $\beta S$ such that $\int_{\beta S} f' \, d\gamma' = \int_F \int_K f(st) \, d\mu(s) \, d\nu(t)$ for $f$ in $CB(S)$. Let $C$ be a nonvoid compact subset of $\beta S - KF$. Let $A = \{f: f$ in $CB(S), \chi_C \leq f' \leq 1\}$. Index the elements of $A$ by $\{f_\lambda\}_{\lambda \in \Lambda}$ so that $\lambda > \eta$ iff $f_\lambda(x) \leq f_\eta(x)$ for each $x$ in $S$. Since the minimum of any pair of elements of $A$ is again in $A$, $\{f_\lambda\}_{\lambda \in \Lambda}$ is a (decreasing) net. For each $\lambda$ in $\Lambda$, let $g_\lambda(t) = \int_K f_\lambda(st) \, d\mu(s)$. Since $\mu \geq 0$, $\{g_\lambda\}_{\lambda \in \Lambda}$ is a decreasing net. By Urysohn's lemma...
$f_\lambda \downarrow 0$ on $KF$ (note that the lemma is applied to the set $C$, $\{x\}$, where $x \in KF$, so that separate continuity suffices). Whence by Dini’s theorem [5, p. 239] $f_\lambda \rightarrow 0$ uniformly on compacta of $KF$. Since $0 \leq f_\lambda \leq 1$ for each $\lambda$, $g_\lambda(t) \downarrow 0$ for each $t \in F$. A similar application of Dini’s theorem implies that $\int_F g_\lambda(t) \, dv(t) \rightarrow 0$. Therefore $0 \leq \gamma'(C) \leq \lim \int_F \int_K f_\lambda(s,t) \, d\mu(s) \, dv(t)$ = 0. Since $KF$ is $\gamma'$-measurable in $\beta S$, $\gamma'(\beta S - KF) = 0$. Therefore $\gamma'(\beta S) = \gamma'(KF)$. Hence by [2, Corollary 4.4, p. 472] there is a unique bounded Radon measure $\gamma$ on $S$ such that $\int_{\beta S} f' \, d\gamma = \int_S f \, d\gamma$ for each $f \in CB(S)$.

REMARK. The above proof remains intact if $KF$ is replaced by any subset $A$ of $S$ containing $KF$ which is measurable with respect to every measure in $M(\beta S)$. In this case the assumption that $KF$ is measurable is not needed.

**THEOREM 2.** Let $S$ be a completely regular separately continuous semigroup. Suppose for each pair of compacta $F, K$ of $S$ there is a subset $A$ of $S$ (depending on $F, K$) such that $A$ is measurable with respect to every measure in $M(\beta S)$. Then the functional $\psi(f) = \int f(st) \, d\mu(s) \, dv(t)$ for $f$ in $CB(S)$ is represented by a unique bounded Radon measure on $S$ for any $\mu$ and $\nu$ in $M(S)$.

**PROOF.** Let $\mu, \nu \geq 0$. For $\varepsilon > 0$, let $F, K$ be a pair of compact subsets of $S$ such that $\mu(S - K) < \varepsilon$, $\nu(S - F) < \varepsilon$.

By Lemma 1, there is a bounded Radon measure $\gamma$ on $S$ such that

$$\int_F \int_K f(st) \, d\mu(s) \, dv(t) = \int f \, d\gamma$$

for each $f$ in $CB(S)$. Let $C$ be a compact subset of $S$ such that $\gamma(S - C) < \varepsilon$. Then $\int f \, d\gamma - \int_C f \, d\gamma$ $\leq \|f\|\varepsilon$. Now

$$\begin{align*}
|\psi(f) - \int_C f \, d\gamma| &\leq \left| \int \int_K f(st) \, d\mu(s) \, dv(t) - \int \int_K f(st) \, d\mu(s) \, dv(t) \right|
+ \left| \int \int_F f(st) \, d\mu(s) \, dv(t) - \int \int_K f(st) \, d\mu(s) \, dv(t) \right|
+ \left| \int \int_C f(st) \, d\mu(s) \, dv(t) - \int f \, d\gamma \right|
\leq \|f\|\mu(S - K)\nu(S) + \|f\|\mu(K)\nu(S - F) + \|f\|\varepsilon
< \|f\|(\nu(S) + \mu(K) + 1)\varepsilon.
\end{align*}$$

By [4, Proposition 4.1 and 2, Lemma 4.5] it follows that there is a unique bounded Radon measure on $S$, which will be denoted by $\mu * \nu$, such that $\psi(f) = \int f \, d\mu * \nu$ for each $f$ in $CB(S)$.

**REMARKS.** (a) For locally compact semigroups $S$ with separately continuous multiplication and for $\mu, \nu$ in $M(S)$, Glicksberg [3] proved that the map $t \rightarrow \int f(st) \, d\mu(s)$ is continuous on $S$ for each $f$ in $CB(S)$). This enabled him to obtain a bounded Radon measure $\mu * \nu$ such that $\int f \, d\mu * \nu = \int \int f(st) \, d\mu(s) \, d\mu(t)$ for each continuous $f$ vanishing at infinity. Later on Pym [6] and recently James C. S. Wong [7] proved that $\int f \, d\mu * \nu = \int \int f(st) \, d\mu(s) \, dv(t)$ for each $f$ in $CB(S)$. The assumption that $S$ is locally compact is essential in their proofs. Since a locally compact $S$ is open in $\beta S$, Theorem 2 implies all of the above.
(b) H. Dzinotyiweyi [1] obtained a similar result for any topological semigroup with jointly continuous multiplication. The joint continuity is essential in his proof. Theorem 2 provides a new proof for his result.

A topological space $X$ is said to be Čech-complete if $X$ is a $G_δ$ is $βX$. Examples of Čech-complete spaces include locally compact spaces and complete metric spaces. The following now is immediate from Theorem 2.

**COROLLARY 3.** Let $S$ be a Čech-complete semigroup with separately continuous multiplication. Then for any $μ, ν$ in $M(S)$, there is a unique measure $μ*ν$ in $M(S)$ such that $\int f dμ*ν = \iint f(st) dμ(s) dν(t)$ for each $f$ in $CB(S)$.

Note that the Glicksberg result [3, Theorem 3.1] implies that $\iint f(st) dμ(s) dν(t)$ $= \iint f(s) dν(t) dμ(s)$ for each $f$ in $CB(S)$ for any topological semigroup $S$ and any $μ, ν$ in $M(S)$. (This follows from the middle inequalities in the proof of Theorem 2 and Glicksberg’s result.)

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**REFERENCES**


