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THE CONVOLUTION OF RADON MEASURES

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ABSTRACT. The convolution of a pair of bounded Radon measures is extended to Čech-complete topological semigroups with separately continuous multiplication.

Let S be a separately continuous topological semigroup, $CB(S)$ the space of all bounded real-valued continuous functions on S and $M(S)$ the set of all bounded real-valued Radon measures on S. For μ in $M(S)$ and f in $CB(S)$, a result of Glicksberg [**3**] implies that the map $t \to \int f(st)\,d\mu(s)$ is continuous when restricted to the compacta of S. Hence it is ν-measurable for any ν in $M(S)$. Let $\psi(f) = \int[\int f(st)\,d\mu(s)]\,d\nu(t)$ for f in $CB(S)$. In this note it is shown that for a large class of semigroups, which includes the class of Čech-complete semigroups, ψ is represented by a unique bounded Radon measure. Denoting this measure by $\mu * \nu$, we have

$$\int f\,d\mu * \nu = \iint f(st)\,d\mu(s)d\nu(t) = \iint f(st)\,d\nu(t)\,d\mu(s)$$

for each f in $CB(S)$. This extends the convolution of a pair of bounded measures to a wide class of topological semigroups with separately continuous multiplication, including those considered in [**1**, **3**, **6**, **7**].

All topological spaces are assumed to be completely regular Hausdorff. βX denotes the Stone-Čech compactification of the topological space X. f' is the unique extension of f in $CB(X)$ to $C(\beta X)$. For the set $A \subset X$, χ_A is the characteristic function of A. $\|\ \|$ denotes the uniform norm.

LEMMA 1. *Let S be a completely regular separately continuous topological semigroup. Let F, K be a pair of compacta of S, such that KF is measurable in βS with respect to every measure in $M(\beta S)$. Then for $\mu, \nu \in M(S)$ the functional, $\psi(f) = \int_F \int_K f(st)\,d\mu(s)\,d\nu(t)$ for f in $CB(S)$, is represented by a unique bounded Radon measure on S.*

PROOF. It is sufficient to prove the lemma for $\mu \geq 0$ and $\nu \geq 0$. Let γ' be a regular finite Borel measure on βS such that $\int_{\beta S} f' d\gamma' = \int_F \int_K f(st)\,d\mu(s)\,d\nu(t)$ for f in $CB(S)$. Let C be a nonvoid compact subset of $\beta S - KF$. Let $A = \{f : f$ in $CB(S)$, $\chi_C \leq f' \leq 1\}$. Index the elements of A by $\{f_\lambda\}_{\lambda \in \Lambda}$ so that $\lambda > \eta$ iff $f_\lambda(x) \leq f_\eta(x)$ for each x in S. Since the minimum of any pair of elements of A is again in A, $\{f_\lambda\}_{\lambda \in \Lambda}$ is a (decreasing) net. For each λ in Λ, let $g_\lambda(t) = \int_K f_\lambda(st)\,d\mu(s)$. Since $\mu \geq 0$, $\{g_\lambda\}_{\lambda \in \Lambda}$ is a decreasing net. By Urysohn's lemma

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$f_\lambda \downarrow 0$ on KF (note that the lemma is applied to the set C, $\{x\}$, where $x \in KF$, so that separate continuity suffices). Whence by Dini's theorem [**5**, p. 239] $f_\lambda \to 0$ uniformly on compacta of KF. Since $0 \le f_\lambda \le 1$ for each λ, $g_\lambda(t) \downarrow 0$ for each t in F. A similar application of Dini's theorem implies that $\int_F g_\lambda(t)\, d\nu(t) \to 0$. Therefore $0 \le \gamma'(C) \le \lim_\lambda \int_F \int_K f_\lambda(st)\, d\mu(s)\, d\nu(t) = 0$. Since KF is γ'-measurable in βS, $\gamma'(\beta S - KF) = 0$. Therefore $\gamma'(\beta S) = \gamma'(KF)$. Hence by [**2**, Corollary 4.4, p. 472] there is a unique bounded Radon measure γ on S such that $\int_{\beta S} f' d\gamma' = \int_S f\, d\gamma$ for each f in $CB(S)$.

REMARK. The above proof remains intact if KF is replaced by any subset A of S containing KF which is measurable with respect to every measure in $M(\beta S)$. In this case the assumption that KF is measurable is not needed.

THEOREM 2. *Let S be a completely regular separately continuous semigroup. Suppose for each pair of compacta F, K of S there is a subset A of S (depending on F, K) such that A is measurable with respect to every measure in $M(\beta S)$. Then the functional $\psi(f) = \iint f(st)\, d\mu(s)\, d\nu(t)$ for f in $CB(S)$ is represented by a unique bounded Radon measure on S for any μ and ν in $M(S)$.*

PROOF. Let μ, $\nu \ge 0$. For $\varepsilon > 0$, let F, K be a pair of compact subsets of S such that $\mu(S - K) < \varepsilon$, $\nu(S - F) < \varepsilon$.

By Lemma 1, there is a bounded Radon measure γ on S such that

$$\int_F \int_K f(st)\, d\mu(s)\, d\nu(t) = \int f\, d\gamma$$

for each f in $CB(S)$. Let C be a compact subset of S such that $\gamma(S - C) < \varepsilon$. Then $|\int f d\gamma - \int_C f\, d\gamma| \le \|f\|\varepsilon$. Now

$$\begin{aligned}\left|\psi(f) - \int_C f\, d\gamma\right| &\le \left|\iint f(st)\, d\mu(s)\, d\nu(t) - \iint_K f(st)\, d\mu(s)\, d\nu(t)\right| \\ &\quad + \left|\iint_K f(st) d\mu(s)\, d\nu(t) - \int_F \int_K f(st)\, d\mu(s)\, d\nu(t)\right| \\ &\quad + \left|\int_F \int_K f(st)\, d\mu(s)\, d\nu(t) - \int_C f\, d\gamma\right| \\ &\le \|f\|\mu(S-K)\nu(S) + \|f\|\mu(K)\nu(S-F) + \|f\|\varepsilon \\ &< \|f\|(\nu(S) + \mu(K) + 1)\varepsilon.\end{aligned}$$

By [**4**, Proposition 4.1 and **2**, Lemma 4.5] it follows that there is a unique bounded Radon measure on S, which will be denoted by $\mu * \nu$, such that $\psi(f) = \int f\, d\mu * \nu$ for each f in $CB(S)$.

REMARKS. (a) For locally compact semigroups S with separately continuous multiplication and for μ, ν in $M(S)$, Glicksberg [**3**] proved that the map $t \to \int f(st)\, d\mu(s)$ is continuous on S for each f in $CB(S)$. This enabled him to obtain a bounded Radon measure $\mu * \nu$ such that $\int f\, d\mu * \nu = \iint f(st)\, d\mu(s)\, d\mu(t)$ for each continuous f vanishing at infinity. Later on Pym [**6**] and recently James C. S. Wong [**7**] proved that $\int f\, d\mu * \nu = \iint f(st)\, d\mu(s)\, d\nu(t)$ for each f in $CB(S)$. The assumption that S is locally compact is essential in their proofs. Since a locally compact S is open in βS, Theorem 2 implies all of the above.

(b) H. Dzinotyiweyi [**1**] obtained a similar result for any topological semigroup with jointly continuous multiplication. The joint continuity is essential in his proof. Theorem 2 provides a new proof for his result.

A topological space X is said to be Čech-complete if X is a G_δ is βX. Examples of Čech-complete spaces include locally compact spaces and complete metric spaces. The following now is immediate from Theorem 2.

COROLLARY 3. *Let S be a Čech-complete semigroup with separately continuous multiplication. Then for any μ, ν in $M(S)$, there is a unique measure $\mu * \nu$ in $M(S)$ such that $\int f \, d\mu * \nu = \iint f(st) \, d\mu(s) \, d\nu(t)$ for each f in $CB(S)$.*

Note that the Glicksberg result [**3**, Theorem 3.1] implies that $\iint f(st) \, d\mu(s) \, d\nu(t) = \iint f(st) \, d\nu(t) \, d\mu(s)$ for each f in $CB(S)$ for any topological semigroup S and any μ, ν in $M(S)$. (This follows from the middle inequalities in the proof of Theorem 2 and Glicksberg's result.)

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