

## APPROXIMATING PA

F. G. J. WIID

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**ABSTRACT.** It is shown that the ring of continuous real-valued functions on a compact Hausdorff space of dimension  $\leq 1$  is a BCS-ring. This result follows from a study of the preservation of the PA and BCS properties under certain ring-theoretic constructions.

Let  $R$  be a commutative ring with identity. Then  $R$  is said to be a PA-ring if each reachable system over  $R$  has its poles arbitrarily assignable, and is said to be a BCS-ring, if each basic submodule of a finitely generated projective  $R$ -module  $M$  contains a rank one projective summand of  $M$  (see below for undefined terms and a more detailed discussion of these ideas). Several recent papers have been concerned with PA- and BCS-rings: e.g. [1, 2, 4, 6, 7]. Among the more important results in this area is the following of Hautus and Sontag.

**THEOREM 0.** *Let  $X$  be a topological manifold with  $C(X)$  the ring of continuous real-valued functions on  $X$ . Then  $C(X)$  is a PA-ring if and only if  $X$  has dimension  $\leq 1$ .*

There is an intriguing formal resemblance between this theorem and the purely algebraic result that a noetherian ring is a BCS-ring (and therefore a PA-ring) provided that it has dimension  $\leq 1$ . (cf. [6, 7]). Given the information that finitely generated projective modules over  $C(X)$ , where  $X$  is a finite simplicial complex, can be represented by projective modules over a noetherian ring of dimension  $\leq \dim(X)$  (cf. [5]), and that projective modules play a key role in the study of PA- and BCS-rings, one might suspect that the relationship between the above results is more than one of formal resemblance. By adapting ideas of Swan [5] we are able to show in this paper that the Hautus-Sontag theorem is indeed a consequence of the algebraic result. Moreover, from our new vantage point it is easy to see how to obtain a generalisation of theorem 0 to the situation where  $X$  is a compact Hausdorff space of dimension  $\leq 1$ .

This paper can also be viewed as a study of the preservation of the PA and BCS properties under certain ring theoretic constructions. One of the essential failures of PA-rings and BCS-rings is that they do not often respect ring-theoretic constructions. In Theorems 1 and 2 below, we wrest some reasonable behaviour from these recalcitrant objects.

Before we state and prove our main results we give the following definitions. A linear system over  $R$  is a pair of maps  $f: U \rightarrow U$ ,  $g: V \rightarrow U$  between finitely

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generated projective  $R$ -modules. We say that  $(f, g)$  is reachable if

$$\bigoplus_n V \xrightarrow{\oplus_n g} \bigoplus_n U \xrightarrow{(f)_i} U$$

is surjective, where  $n = \text{rank}(U)$ , and pole assignable, if, given  $(a_1, \dots, a_n) \in R^n$ , there is a map  $k: U \rightarrow V$  such that  $a_1, \dots, a_n$  are the eigenvalues of  $f + gk$ .  $R$  is PA if every reachable system over  $R$  is pole assignable.  $R$  is PAF if every system with  $U$  and  $V$  free is pole assignable.

It can be shown that  $R$  is PA if and only if for every reachable system  $(f, g)$  over  $R$  the following condition holds:  $\text{Im}(g)$  contains a rank one projective summand of  $\text{Dmn}(f)$ .

A ring  $R$  is a BCS-ring if every basic submodule of a finitely generated projective  $R$ -module  $P$  contains a rank one summand of  $P$ . It is fairly simple to show that  $R$  is a BCS-ring if and only if the following condition is satisfied: if  $G$  is an  $R$ -matrix having unit content then there is a matrix  $V$  such that  $GV$  is a  $*$ -matrix (i.e.  $GV$  has unit content and all  $2 \times 2$  minors of  $GV$  are zero) (cf. [1, 6]).

In the sequel  $f: A \rightarrow B$  is a morphism of rings such that:

- (1)  $B$  is a topological ring,  $B^*$  (the units of  $B$ ) is open in  $B$  and  $u \rightarrow u^{-1}$  is continuous on  $B^*$ ,
- (2)  $f(A)$  is dense in  $B$ ,
- (3) if  $f(a)$  is sufficiently near 1 in  $B$ , then  $a \in A^*$ .

Swan (5) also introduces the technical condition (SBI). We shall refrain from explicating this condition—it ensures that each finitely generated projective  $B$ -module is extended from  $A$ .

There are two theorems.

**THEOREM 1.** *Let  $f: A \rightarrow B$  satisfy (1)–(3). If  $A$  is a BCS-ring, then  $B$  is a BCS-ring. If, in addition,  $f$  also satisfies (SBI), then  $B$  is a PA-ring if  $A$  is a PA-ring.*

**THEOREM 2.** *Let  $\{f_{\alpha\mu}: A_\alpha \rightarrow A_\mu\}_{\alpha \leq \mu}$  be a directed system of rings. If each  $A_\alpha$  is a BCS-ring (resp. a PAF-ring or a PA-ring) then so is  $\varinjlim A_\alpha$ .*

**PROOF OF THEOREM 1.** The first assertion is proved as follows. Let  $G = (c_{ij}): B^m \rightarrow B^n$  be a matrix having unit content, say  $1 = \sum_{i,j} r_{ij}c_{ij}$  for some  $r_{ij} \in B$ . By (2), there exist  $s_{ij}, d_{ij} \in A$  such that  $f(s_{ij})$  is arbitrarily close to  $r_{ij}$  and  $f(d_{ij})$  is arbitrarily close to  $c_{ij}$ . Thus by (1), we can choose  $s_{ij}, d_{ij}$  such that  $f(\sum_{i,j} s_{ij}d_{ij}) = \sum_{i,j} f(s_{ij})f(d_{ij})$  is arbitrarily close to  $\sum r_{ij}c_{ij} = 1$ . Thus we can bring  $f(\sum_{i,j} s_{ij}d_{ij})$  sufficiently close to 1 to ensure (by (3)) that  $\sum_{i,j} s_{ij}d_{ij}$  is a unit. Therefore  $G' = (d_{ij})$  has unit content. Since  $A$  is a BCS ring, there is a rank one projective summand  $P$  in the image of  $G'$ : that is, denoting by  $\Pi$  the canonical projection, the following sequence is exact

$$A^m \xrightarrow{\Pi \cdot G'} P \rightarrow 0.$$

Then the sequence

$$B^m \xrightarrow{(\Pi \otimes 1_B) \cdot f(G')} P \otimes B \rightarrow 0$$

is exact and the map  $(\Pi \otimes 1_B) \cdot G$  is sufficiently close to  $(\Pi \otimes 1_B) \cdot f(G')$ . Thus, by [5, Lemma 1.3], the map  $\Pi \otimes 1_B \cdot G$  is surjective and this gives a rank one projective summand in the image of  $G$ . It follows that  $B$  is a BCS-ring.

The proof of the second assertion is an adaptation of the above proof: one uses (SBI) to show that each reachable  $B$ -system can be approximated arbitrarily closely by a reachable  $A$ -system and then one reasons as above.

**PROOF OF THEOREM 2.** A proof of the BCS portion of the theorem goes as follows. Let  $G \in M_{n \times m}(A)$  have unit content. Write  $G = (a_{ij})$ , then there exist  $b_{ij} \in A$  so that  $\sum_{i,j} a_{ij}b_{ij} = 1$ . If  $f_\alpha: F_\alpha \rightarrow \varinjlim F_\alpha$  are the universal mappings, we can find  $\lambda \in I$  such that there exist  $c_{ij}, d_{ij} \in A_\lambda$  so that  $f_\lambda(c_{ij}) = a_{ij}$  and  $f_\lambda(d_{ij}) = b_{ij}$ . Then  $f_\lambda((\sum_{i,j} c_{ij}d_{ij}) - 1) = 0$ , so there exists a  $\mu \geq \lambda$  so that  $f_{\lambda\mu}((\sum_{i,j} c_{ij}d_{ij}) - 1) = 0$ . Then  $\sum_{i,j} f_{\lambda\mu}(c_{ij})f_{\lambda\mu}(d_{ij}) = 1$ . Let  $G_\mu = (f_{\lambda\mu}(c_{ij})) \in M_{n \times m}(A_\mu)$ . Then  $G_\mu$  has unit content, and  $f_\mu(G_\mu) = G$ . Since  $A_\mu$  is BCS, there exists a matrix  $B_\mu \in M_{m \times k}(A_\mu)$  such that  $G_\mu B_\mu$  is a  $*$ -matrix. Let  $B = f_\mu(B_\mu)$ . Then  $f_\mu(G_\mu B_\mu) = GB$  is a  $*$ -matrix over  $A$ .

The PAF and CAF proofs are completely analogous, but the PA assertion requires some formalism involving direct limits of categories. This is also easily adapted from [5, pp. 214–215].

These results have the following

**COROLLARY.** *Let  $X$  be a compact Hausdorff space of dimension  $\leq 1$ . Then  $C(X)$  is a BCS-ring.*

**PROOF.** By [3, p. 278],  $X = \lim X_\alpha$ , where  $\{X_\alpha\}$  is an inverse system of polyhedra of dimension  $\leq 1$ . Then by [5, p. 215], the ring morphism  $f: \varinjlim C(X_\alpha) \rightarrow C(X)$  satisfies (1)–(3) and so by Theorems 1 and 2 we would be finished if we knew that  $C(X_\alpha)$  were a BCS-ring for each  $\alpha$ . But by [5, Theorem 6.3], for each  $\alpha$  there exist a noetherian subring  $B_\alpha$  of  $C(X_\alpha)$  having dimension  $\leq 1$ . Moreover, the inclusion  $B_\alpha \subset C(X_\alpha)$  satisfies (1)–(3). By [6 or 7],  $B_\alpha$  is a BCS-ring and another application of Theorem 1 completes the proof.

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NATIONAL RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, CSIR, PO BOX 395,  
PRETORIA 0001, SOUTH AFRICA

*Current address:* Department of Mathematics, University of the Witwatersrand, No. 1, Jan  
Smuts Avenue, Johannesburg 2000, South Africa