

## AUTOMATIC CONTINUITY OF CONCAVE FUNCTIONS

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**ABSTRACT.** A necessary and sufficient condition is given that a semicontinuous, nonnegative, concave function on a finite dimensional closed convex set  $X$  necessarily be continuous at a point  $x_0 \in X$ . Application of this criterion at all points of  $X$  yields a characterization, due to Gale, Klee and Rockafellar, of convex polyhedra in terms of continuity of their convex functions.

Let  $V$  be a real vector space of dimension  $n < \infty$ . Let  $X \subseteq V$  be a closed convex body. Let  $\phi$  be a concave, nonnegative function on  $X$ . (Recall  $\phi$  is concave if  $-\phi$  is convex.) Define  $G^-(\phi)$ , the *subgraph* of  $\phi$ , as the subset of  $V \times \mathbf{R}$  specified by

$$(1) \quad G^-(\phi) = \{(x, t) : x \in X, 0 \leq t \leq \phi(x)\}.$$

If  $\phi$  is not identically zero then  $G^-(\phi)$  will be a convex body in  $V \times \mathbf{R}$ . We call  $\phi$  *semicontinuous* if  $G^-(\phi)$  is a closed subset of  $V \times \mathbf{R}$ . (This is usually called upper semicontinuity; since lower semicontinuity is not very important here, we let the "upper" be understood implicitly.) Observe that this is equivalent to the superlevel sets

$$(2) \quad L^+(\phi, s) = \{x \in X : s \leq \phi(x)\}, \quad s \geq 0,$$

being closed. Observe also that the  $L^+(\phi, s)$  are convex.

We say  $X$  is *polyhedral* if it is specified by a finite number of linear inequalities

$$(3) \quad X = \{v \in V : \lambda_i(v) \leq b_i, \lambda_i \in V^*, b_i \in \mathbf{R}, 1 \leq i \leq m\}.$$

In [GKR] (see also [R, §10]) it is shown that if  $\phi$  is a nonnegative, concave, semicontinuous function on  $X$ , and  $X$  is polyhedral, then  $\phi$  is in fact continuous. (Actually, in [GKR], convex functions are considered; but concave and convex are interchangeable here.) The purpose of this note is to refine the result by giving a pointwise criterion for automatic continuity. If our condition holds at all points of a convex set  $X$ , then  $X$  is close to being polyhedral. (More precisely it is boundedly polyhedral in the sense of [GKR]; see Proposition 3.)

With  $X$  as above, suppose that for some  $t > 0$  we have a closed convex set of  $Y \subseteq V \times [0, t]$  such that

$$(4) \quad \begin{aligned} (a) & Y \cap (V \times \{0\}) = X \times \{0\} \\ (b) & \text{If } (x, r) \in Y, \text{ then } (x, r') \in Y \text{ for } 0 \leq r' \leq r. \end{aligned}$$

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Then the recipe

(c)  $\phi_Y(x) = \max\{r: (x, r) \in Y\}$ ,  $x \in X$ , defines a concave nonnegative function on  $X$ , and

(d)  $Y = G^-(\phi_Y)$ .

Denote by  $\text{SCNC}(X)$  the set of semicontinuous, nonnegative concave functions on  $X$ . It is straightforward to check that the sum of two functions in  $\text{SCNC}(X)$  is again in  $\text{SCNC}(X)$ . Also a positive scalar multiple of an element in  $\text{SCNC}(X)$  is again an element. Thus  $\text{SCNC}(X)$  is a cone in the space of all real-valued functions on  $X$ . Also given a family  $\{\phi_i\}_{i \in I}$  of functions in  $\text{SCNC}(X)$  (the index set  $I$  may be infinite), we may form their infimum

$$(5)(a) \quad \inf\{\phi_i\}(x) = \inf\{\phi_i(x) : i \in I\}, \quad x \in X.$$

It is easy to see that  $\inf\{\phi_i\}$  is concave and nonnegative. We also clearly have

$$(b) \quad G^-(\inf\{\phi_i\}) = \bigcap_i G^-(\phi_i)$$

so that  $\inf\{\phi_i\}$  again belongs to  $\text{SCNC}(X)$ .

Let  $Z \subseteq X$  be an arbitrary subset of  $X$ , and let  $f$  be an arbitrary real-valued function on  $Z$ . Consider the set of  $\phi$  in  $\text{SCNC}(X)$  such that  $\phi$  dominates  $f$  on  $Z$  (i.e.,  $\phi(z) \geq f(z)$  for all  $z \in Z$ ). Evidently, the infimum of such  $\phi$  will again dominate  $f$ . Thus if there are any elements of  $\text{SCNC}(X)$  dominating  $f$  on  $Z$ , there is a minimum one. In particular, given a point  $x_0 \in X$ , there is a minimum element of  $\text{SCNC}(X)$  taking the value 1 at  $x_0$ .

Proposition 1: Given  $x_0 \in X$ , define a function  $E_X(x_0, x)$  on  $X$  by

$$(6) \quad E_X(x_0, x) = \sup\{(t - 1)/t : x_0 + t(x - x_0) \in X\}, \quad x \in X, \\ = \sup\{s \in [0, 1] : x = sx_0 + (1 - s)z \text{ for some } z \in X\}.$$

Then  $E_X(x_0, \cdot)$  is the minimum among elements of  $\text{SCNC}(X)$  taking the value 1 at  $x_0$ .

REMARK. In pictorial terms we may describe the (closure of the) graph of  $E_X(x_0, \cdot)$  as the surface of the cone with base  $X \times \{0\}$  and vertex  $(x_0, 1)$ .

PROOF. In  $V \times \mathbf{R}$ , let  $C(X, x_0)$  denote the closed convex hull of the points  $(x, 0)$ ,  $x \in X$ , and the point  $(x_0, 1)$ . Since  $X$  is convex, the convex hull of  $X \times \{0\}$  and  $(x_0, 1)$  is the set  $\{(sx_0 + (1 - s)y, s) : y \in X, 0 \leq s \leq 1\}$  and  $C(X, x_0)$  will be the closure of this set. Suppose  $x \neq x_0$ , and

$$(x, r) = (sx_0 + (1 - s)y, s).$$

Then  $r = s < 1$ , and

$$y = x_0 + (1 - s)^{-1}(x - x_0)$$

belongs to  $X$ . Setting  $t = (1 - s)^{-1}$  we have

$$r = s = 1 - t^{-1} = (t - 1)/t.$$

From the convexity of  $X$  it is clear that if  $(x, r)$  is in  $C(X, x_0)$ , then so is  $(x, r')$  for  $0 \leq r' \leq r$ . Hence  $C(X, x_0)$  satisfies conditions (4)(a)(b), and comparing (4)(c)(d) with (6) shows

$$C(X, x_0) = G^-(E_X(x_0, \cdot)).$$

Furthermore, if  $\phi$  is any function in  $\text{SCNC}(X)$  such that  $\phi(x_0) \geq 1$ , then obviously  $G^-(\phi) \supseteq C(X, x_0)$ , whence  $\phi(x) \geq E_X(x_0, x)$ . This proves Proposition 1.

Given a point  $x_0$  in  $X$ , we say  $X$  is *conical at  $x_0$*  if there exist

- (i) a neighborhood  $U$  of  $x_0$  in  $V$  and,
- (ii) a closed convex cone  $C \subseteq V$ ,

such that

$$(7) \quad X \cap U = (C + x_0) \cap U.$$

That is, near  $x_0$ , the set  $X$  looks like a translated cone. Note that  $C$  need not be a proper, also called pointed, cone. In particular, we could take  $C = V$ . Thus  $X$  is conical at all of its interior points.

**PROPOSITION 2.** (a) *If  $E_X(x_0, \cdot)$  (cf. formula (6)) is continuous at  $x_0$ , then all functions in  $\text{SCNC}(X)$  are continuous at  $x_0$ .*

(b) *The function  $E_X(x_0, \cdot)$  is continuous at  $x_0$  if and only if  $X$  is conical at  $x_0$ .*

**PROOF.** (a) Suppose  $E_X(x_0, \cdot)$  is continuous at  $x_0$ . Then given  $\varepsilon > 0$ , there is a neighborhood  $U$  of  $x_0$  such that  $E_X(x_0, x) > 1 - \varepsilon$  for  $x \in U \cap X$ . Consider  $\phi \in \text{SCNC}(X)$ . By semicontinuity the superlevel set  $L^+(\phi, \phi(x_0) + \varepsilon)$  (cf. (2)) is closed, and since it does not contain  $x_0$ , the set  $U'' = V - L^+(\phi, \phi(x_0) + \varepsilon)$  is a neighborhood of  $x_0$ . If  $\phi(x_0) = 0$ , then since  $\phi \geq 0$ , we see  $|\phi(x) - \phi(x_0)| < \varepsilon$  on  $U'' \cap X$ , so  $\phi$  is continuous at  $x_0$ . If  $\phi(x_0) > 0$ , then it suffices to show  $\phi(x)/\phi(x_0)$  is continuous at  $x_0$ . Hence we may assume  $\phi(x_0) = 1$ . Then on the neighborhood  $U \cap U'' \cap X$  of  $x$  in  $X$  we have  $1 + \varepsilon > \phi(x) > E_X(x_0, x) > 1 - \varepsilon$ . Hence again  $\phi$  is continuous at  $x_0$ .

(b) Let  $U$  be an open convex neighborhood of the origin in  $V$ , with compact closure  $\bar{U}$ . Then any neighborhood of  $x_0$  contains a set of the form  $x_0 + \delta U$  for a suitably small number  $\delta > 0$ . Let  $\partial U = \bar{U} - U$  be the boundary of  $U$ . If  $C$  is any closed convex cone in  $V$  then we have

$$C = \bigcup_{s \geq 0} s(C \cap \partial U).$$

Suppose  $E_X(x_0, \cdot)$  is continuous at  $x_0$ . Then we can find  $\delta > 0$  such that  $E_X(x_0, x) > \frac{1}{2}$  for  $x \in (x_0 + \delta U) \cap X$ . Set

$$B = (x_0 + \delta(\partial U)) \cap X, \quad C = \bigcup_{s \geq 0} s(B - x_0).$$

Then  $C$  is a cone (a union of rays), and clearly

$$(8) \quad (C + x_0) \cap (x_0 + \delta U) \subseteq X \cap (x_0 + \delta U).$$

For if  $x \in C + x_0$ , then  $x = x_0 + sb$ ,  $b \in B$ ,  $s \geq 0$ ; and if  $x \in x_0 + \delta U$ , then  $s < 1$ . Hence  $x = (1 - s)x_0 + s(x_0 + b) \in X$ , since  $X$  is convex. I claim that in fact the inclusion (8) is an equality. To verify this, consider a point  $v$  in  $(x_0 + \delta U) \cap X$ . Assume  $y \neq x_0$ . For suitable  $t \geq 1$  the point  $z = x_0 + t(y - x_0)$  will be in  $x_0 + \delta(\partial U)$ . If we show  $z \in X$ , the claim will be established. Suppose  $z \notin X$ . Since  $X$  is closed and convex, there is a number  $a$ ,  $0 < a < 1$  such that the points  $z_r = x_0 + r(z - x_0)$  are in  $X$  for  $r \leq a$ , and are not in  $X$  for  $r > a$ . We see then that  $E_X(x_0, z_a) = 0$ . But since clearly  $z_a \in X \cap (x_0 + \delta U)$ , this contradicts our choice of  $\delta$ . Thus inclusion (8) is an equality, and  $X$  is conical at  $x_0$ .

Conversely, suppose  $X$  is conical at  $x_0$ . Let  $U$  be a convex neighborhood of the origin, and  $C$  a closed convex cone such that

$$(9) \quad (a) \quad (x_0 + U) \cap X = x_0 + (C \cap U).$$

Then for  $0 < a \leq 1$ , the set

$$(b) \quad U'_a = (x_0 + aU) \cap X = x_0 + a(C \cap U)$$

will be a neighborhood of  $x_0$  in  $X$ . Taking  $t$  in formula (6) to be  $\frac{1}{a}$  we see that  $E_X(x_0, x) \geq 1 - a$  if  $x \in U'_a$ . Hence  $E_X(x_0, \cdot)$  is continuous at  $x_0$ . This proves Proposition 2.

The connection of the above two results with automatic continuity is provided by the following result. Given a point  $x_0 \in X$ , we say  $X$  is *polyhedral at  $x_0$*  if there is a polyhedral closed convex subset  $P_{x_0} \subseteq X$  such that  $P_{x_0}$  contains a neighborhood of  $x_0$  in  $X$ . We say  $S$  is *locally polyhedral* if  $X$  is polyhedral at each of its points. We say  $X$  is *semilocally polyhedral* or *boundedly polyhedral* if any compact subset  $C \subseteq X$  is contained in a polyhedral subset  $P \subseteq X$ .

This definition may seem superficially different from the definition of boundedly polyhedral in [GKR, p. 867], but it is easily seen to be equivalent.

PROPOSITION 3. *The following are equivalent:*

- (i)  $X$  is conical at each of its points.
- (ii)  $X$  is locally polyhedral.
- (iii)  $X$  is semilocally polyhedral.
- (iv) All  $\phi \in \text{SCNC}(X)$  are continuous.

REMARKS. (a) The implication (ii)  $\Rightarrow$  (i) has a local version: if  $X$  is polyhedral at  $x_0$ , then  $X$  is conical at  $x_0$ ; the implication (i)  $\Rightarrow$  (ii) has no such local version.

(b) The implication (i)  $\Rightarrow$  (ii) can be deduced from [K] (see especially Theorems 4.1 and 4.7), but we give a short proof.

(c) The equivalence (iii)  $\Leftrightarrow$  (iv) amounts more or less to the equivalence  $(BP) \Leftrightarrow S$  of Theorem 2 of [GKR].

PROOF. The implication (iii)  $\Rightarrow$  (ii) is trivial. The implication (ii)  $\Rightarrow$  (i) is routine; we omit its proof. The equivalence (i)  $\Leftrightarrow$  (iv) follows from Proposition 2. The implication (ii)  $\Rightarrow$  (iii) is Proposition 2.17 of [K].

We prove (i)  $\Rightarrow$  (ii) by induction on  $\dim X = \dim V$ . If  $\dim V = 2$ , it is immediate since closed convex cones in 2-space are polyhedral. It follows directly from the definitions that if  $X$  is conical at every point, and  $A \subseteq V$  is an affine subspace, then  $X \cap A$  is conical at every point. Hence, if  $\dim A < \dim V$  we may assume  $A \cap X$  is locally polyhedral. If the neighborhood  $U$  in the proof of Proposition (2b) (see inclusion (8)) is chosen so that its closure  $\bar{U}$  is polyhedral, then (using (ii)  $\Rightarrow$  (iii)) we see that the intersection of  $X$  with each codimension one face of  $x_0 + \delta\bar{U}$  will be polyhedral. Hence the set  $B$  is polyhedral (in the sense that it is a finite union of convex polyhedra; it may not be convex), and in particular has a finite number of extreme points. By (8) (which, we recall, is an equality, not just an inclusion) we see that  $X \cap (x_0 + \delta\bar{U})$  is the convex hull of (the extreme points of)  $B$  and of  $x_0$ , and so is polyhedral.

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