AUTOMATIC CONTINUITY OF CONCAVE FUNCTIONS

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ABSTRACT. A necessary and sufficient condition is given that a semicontinuous, nonnegative, concave function on a finite dimensional closed convex set $X$ necessarily be continuous at a point $x_0 \in X$. Application of this criterion at all points of $X$ yields a characterization, due to Gale, Klee and Rockafellar, of convex polyhedra in terms of continuity of their convex functions.

Let $V$ be a real vector space of dimension $n < \infty$. Let $X \subseteq V$ be a closed convex body. Let $\phi$ be a concave, nonnegative function on $X$. (Recall $\phi$ is concave if $-\phi$ is convex.) Define $G^- (\phi)$, the subgraph of $\phi$, as the subset of $V \times \mathbb{R}$ specified by
\[
G^- (\phi) = \{(x, t) : x \in X, \ 0 \leq t \leq \phi(x)\}.
\]
If $\phi$ is not identically zero then $G^- (\phi)$ will be a convex body in $V \times \mathbb{R}$. We call $\phi$ semicontinuous if $G^- (\phi)$ is a closed subset of $V \times \mathbb{R}$. (This is usually called upper semicontinuity; since lower semicontinuity is not very important here, we let the “upper” be understood implicitly.) Observe that this is equivalent to the superlevel sets\[
L^+ (\phi, s) = \{x \in X : s \leq \phi(x)\}, \quad s \geq 0,
\]
being closed. Observe also that the $L^+ (\phi, s)$ are convex.

We say $X$ is polyhedral if it is specified by a finite number of linear inequalities
\[
X = \{v \in V : \lambda_i (v) \leq b_i, \lambda_i \in V^*, b_i \in \mathbb{R}, 1 \leq i \leq m\}.
\]

In [GKR] (see also [R, §10]) it is shown that if $\phi$ is a nonnegative, concave, semicontinuous function on $X$, and $X$ is polyhedral, then $\phi$ is in fact continuous. (Actually, in [GKR], convex functions are considered; but concave and convex are interchangeable here.) The purpose of this note is to refine the result by giving a pointwise criterion for automatic continuity. If our condition holds at all points of a convex set $X$, then $X$ is close to being polyhedral. (More precisely it is boundedly polyhedral in the sense of [GKR]; see Proposition 3.)

With $X$ as above, suppose that for some $t > 0$ we have a closed convex set of $Y \subseteq V \times [0, t]$ such that
\[
Y \cap (V \times \{0\}) = X \times \{0\}
\]
(b) If $(x, r) \in Y$, then $(x, r') \in Y$ for $0 \leq r' \leq r$.

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Then the recipe
(c) \( \phi_Y(x) = \max\{r: (x, r) \in Y\} \), \( x \in X \), defines a concave nonnegative function
on \( X \), and
(d) \( Y = G^{-}(\phi_Y) \).

Denote by \( \text{SCNC}(X) \) the set of semicontinuous, nonnegative concave functions
on \( X \). It is straightforward to check that the sum of two functions in \( \text{SCNC}(X) \) is
again in \( \text{SCNC}(X) \). Also a positive scalar multiple of an element in \( \text{SCNC}(X) \) is
again an element. Thus \( \text{SCNC}(X) \) is a cone in the space of all real-valued functions
on \( X \). Also given a family \( \{\phi_i\}_{i \in I} \) of functions in \( \text{SCNC}(X) \) (the index set \( I \) may
be infinite), we may form their infimum

\[
\inf\{\phi_i\}(x) = \inf\{\phi_i(x): i \in I\}, \quad x \in X.
\]

It is easy to see that \( \inf\{\phi_i\} \) is concave and nonnegative. We also clearly have

\[
G^{-}(\inf\{\phi_i\}) = \bigcap_i G^{-}(\phi_i)
\]

so that \( \inf\{\phi_i\} \) again belongs to \( \text{SCNC}(X) \).

Let \( Z \subseteq X \) be an arbitrary subset of \( X \), and let \( f \) be an arbitrary real-valued
function on \( Z \). Consider the set of \( \phi \) in \( \text{SCNC}(X) \) such that \( \phi \) dominates \( f \) on
\( Z \) (i.e., \( \phi(z) \geq f(z) \) for all \( z \in Z \)). Evidently, the infimum of such \( \phi \) will again
dominate \( f \). Thus if there are any elements of \( \text{SCNC}(X) \) dominating \( f \) on \( Z \), there
is a minimum one. In particular, given a point \( x_0 \in X \), there is a minimum element
of \( \text{SCNC}(X) \) taking the value 1 at \( x_0 \).

**Proposition 1:** Given \( x_0 \in X \), define a function \( E_X(x_0, x) \) on \( X \) by

\[
E_X(x_0, x) = \sup\{ (t-1)/t: x_0 + t(x-x_0) \in X \}, \quad x \in X,
\]

\[
= \sup\{ s \in [0,1]: x = sx_0 + (1-s)z \text{ for some } z \in X \}.
\]

Then \( E_X(x_0, \cdot) \) is the minimum among elements of \( \text{SCNC}(X) \) taking the value 1 at \( x_0 \).

**REMARK.** In pictorial terms we may describe the (closure of the) graph of
\( E_X(x_0, \cdot) \) as the surface of the cone with base \( X \times \{0\} \) and vertex \((x_0, 1)\).

**PROOF.** In \( V \times \mathbb{R} \), let \( C(X, x_0) \) denote the closed convex hull of the points \((x, 0)\), \( x \in X \), and the point \((x_0, 1)\). Since \( X \) is convex, the convex hull of \( X \times \{0\} \) and
\((x_0, 1)\) is the set \( \{(sx_0 + (1-s)y, s): y \in X, 0 \leq s \leq 1\} \) and \( C(X, x_0) \) will be the
closure of this set. Suppose \( x \neq x_0 \), and

\[
(x, r) = (sx_0 + (1-s)y, s).
\]

Then \( r = s < 1 \), and

\[
y = x_0 + (1-s)^{-1}(x-x_0)
\]

belongs to \( X \). Setting \( t = (1-s)^{-1} \) we have
\[
r = s = 1 - t^{-1} = (t-1)/t.
\]

From the convexity of \( X \) it is clear that if \((x, r)\) is in \( C(X, x_0) \), then so is \((x, r')\) for
\( 0 \leq r' \leq r \). Hence \( C(X, x_0) \) satisfies conditions (4)(a)(b), and comparing (4)(c)(d)
with (6) shows

\[
C(X, x_0) = G^{-}(E_X(x_0, \cdot)).
\]

Furthermore, if \( \phi \) is any function in \( \text{SCNC}(X) \) such that \( \phi(x_0) \geq 1 \), then obviously
\( G^{-}(\phi) \supseteq C(X, x_0) \), whence \( \phi(x) \geq E_X(x_0, x) \). This proves Proposition 1.
Given a point $x_0$ in $X$, we say $X$ is conical at $x_0$ if there exist
(i) a neighborhood $U$ of $x_0$ in $V$ and,
(ii) a closed convex cone $C \subseteq V$,
such that
\begin{equation}
X \cap U = (C + x_0) \cap U.
\end{equation}
That is, near $x_0$, the set $X$ looks like a translated cone. Note that $C$ need not be
a proper, also called pointed, cone. In particular, we could take $C = V$. Thus $X$ is
conical at all of its interior points.

PROPOSITION 2. (a) If $E_X(x_0, \cdot)$ (cf. formula (6)) is continuous at $x_0$, then
all functions in $\text{SCNC}(X)$ are continuous at $x_0$.
(b) The function $E_X(x_0, \cdot)$ is continuous at $x_0$ if and only if $X$ is conical at $x_0$.

PROOF. (a) Suppose $E_X(x_0, \cdot)$ is continuous at $x_0$. Then given $\epsilon > 0$, there
is a neighborhood $U$ of $x_0$ such that $E_X(x_0, x) > 1 - \epsilon$ for $x \in U \cap X$. Consider
$\phi \in \text{SCNC}(X)$. By semicontinuity the superlevel set $L^+(\phi, x_0 + \epsilon)$ (cf. (2)) is
closed, and since it does not contain $x_0$, the set $U'' = V - L^+(\phi, x_0 + \epsilon)$ is a
neighborhood of $x_0$. If $\phi(x_0) = 0$, then since $\phi \geq 0$, we see $|\phi(x) - \phi(x_0)| < \epsilon$ on
$U'' \cap X$, so $\phi$ is continuous at $x_0$. If $\phi(x_0) > 0$, then it suffices to show $\phi(x)/\phi(x_0)$
is continuous at $x_0$. Hence we may assume $\phi(x_0) = 1$. Then on the neighborhood
$U \cap U'' \cap X$ of $x$ in $X$ we have $1 + \epsilon > \phi(x) > E_X(x_0, x) > 1 - \epsilon$. Hence again $\phi$ is
continuous at $x_0$.

(b) Let $U$ be an open convex neighborhood of the origin in $V$, with compact
closure $\overline{U}$. Then any neighborhood of $x_0$ contains a set of the form $x_0 + \delta U$ for a
suitably small number $\delta > 0$. Let $\partial U = \overline{U} - U$ be the boundary of $U$. If $C$ is any
closed convex cone in $V$ then we have
\begin{equation}
C = \bigcup_{s \geq 0} s(C \cap \partial U).
\end{equation}
Suppose $E_X(x_0, \cdot)$ is continuous at $x_0$. Then we can find $\delta > 0$ such that
$E_X(x_0, x) > \frac{1}{2}$ for $x \in (x_0 + \delta U) \cap X$. Set
\begin{equation}
B = (x_0 + \delta(\partial U)) \cap X, \quad C = \bigcup_{s \geq 0} s(B - x_0).
\end{equation}
Then $C$ is a cone (a union of rays), and clearly
\begin{equation}
(C + x_0) \cap (x_0 + \delta U) \subseteq X \cap (x_0 + \delta U).
\end{equation}
For if $x \in C + x_0$, then $x = x_0 + sb$, $b \in B$, $s \geq 0$; and if $x \in x_0 + \delta U$, then $s < 1$.
Hence $x = (1-s)x_0 + s(x_0 + b) \in X$, since $X$ is convex. I claim that in fact the
inclusion (8) is an equality. To verify this, consider a point $v$ in $(x_0 + \delta U) \cap X$.
Assume $y \neq x_0$. For suitable $t \geq 1$ the point $z = x_0 + t(y - x_0)$ will be in $x_0 + \delta(\partial U)$.
If we show $z \in X$, the claim will be established. Suppose $z \notin X$. Since $X$ is closed
and convex, there is a number $a$, $0 < a < 1$ such that the points $z_r = x_0 + r(z - x_0)$
are in $X$ for $r \leq a$, and are not in $X$ for $r > a$. We see then that $E_X(x_0, z_a) = 0$.
But since clearly $z_a \in X \cap (x_0 + \delta U)$, this contradicts our choice of $\delta$. Thus inclusion
(8) is an equality, and $X$ is conical at $x_0$. 

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Conversely, suppose $X$ is conical at $x_0$. Let $U$ be a convex neighborhood of the origin, and $C$ a closed convex cone such that

\[(x_0 + U) \cap X = x_0 + (C \cap U) \tag{9}\]

Then for $0 < a \leq 1$, the set

\[U'_a = (x_0 + aU) \cap X = x_0 + a(C \cap U)\]

will be a neighborhood of $x_0$ in $X$. Taking $t$ in formula (6) to be $\frac{1}{a}$, we see that $E_X(x_0, x) \geq 1 - a$ if $x \in U'_a$. Hence $E_X(x_0, \cdot)$ is continuous at $x_0$. This proves Proposition 2.

The connection of the above two results with automatic continuity is provided by the following result. Given a point $x_0 \in X$, we say $X$ is polyhedral at $x_0$ if there is a polyhedral closed convex subset $P_{x_0} \subseteq X$ such that $P_{x_0}$ contains a neighborhood of $x_0$ in $X$. We say $S$ is locally polyhedral if $X$ is polyhedral at each of its points. We say $X$ is semilocus polyhedral or boundedly polyhedral if any compact subset $C \subseteq X$ is contained in a polyhedral subset $P \subseteq X$.

This definition may seem superficially different from the definition of boundedly polyhedral in [GKR, p. 867], but it is easily seen to be equivalent.

**PROPOSITION 3.** The following are equivalent:

(i) $X$ is conical at each of its points.

(ii) $X$ is locally polyhedral.

(iii) $X$ is semilocally polyhedral.

(iv) All $\phi \in SCNC(X)$ are continuous.

**REMARKS.** (a) The implication (ii) $\Rightarrow$ (i) has a local version: if $X$ is polyhedral at $x_0$, then $X$ is conical at $x_0$; the implication (i) $\Rightarrow$ (ii) has no such local version.

(b) The implication (i) $\Rightarrow$ (ii) can be deduced from [K] (see especially Theorems 4.1 and 4.7), but we give a short proof.

(c) The equivalence (iii) $\Leftrightarrow$ (iv) amounts more or less to the equivalence $(BP) \Leftrightarrow S$ of Theorem 2 of [GKR].

**PROOF.** The implication (iii) $\Rightarrow$ (ii) is trivial. The implication (ii) $\Rightarrow$ (i) is routine; we omit its proof. The equivalence (i) $\Leftrightarrow$ (iv) follows from Proposition 2. The implication (ii) $\Rightarrow$ (iii) is Proposition 2.17 of [K].

We prove (i) $\Rightarrow$ (ii) by induction on $\dim X = \dim V$. If $\dim V = 2$, it is immediate since closed convex cones in 2-space are polyhedral. It follows directly from the definitions that if $X$ is conical at every point, and $A \subseteq V$ is an affine subspace, then $X \cap A$ is conical at every point. Hence, if $\dim A < \dim V$ we may assume $A \cap X$ is locally polyhedral. If the neighborhood $U$ in the proof of Proposition (2b) (see inclusion (8)) is chosen so that its closure $\overline{U}$ is polyhedral, then (using (ii) $\Rightarrow$ (iii)) we see that the intersection of $X$ with each codimension one face of $x_0 + \delta \overline{U}$ will be polyhedral. Hence the set $B$ is polyhedral (in the sense that it is a finite union of convex polyhedra; it may not be convex), and in particular has a finite number of extreme points. By (8) (which, we recall, is an equality, not just an inclusion) we see that $X \cap (x_0 + \delta \overline{U})$ is the convex hull of (the extreme points of) $B$ and of $x_0$, and so is polyhedral.
REFERENCES


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