CLOSED CONVEX HYPERSURFACES
WITH CURVATURE RESTRICTIONS

ROLF SCHNEIDER

(Communicated by David G. Ebin)


1. Introduction. For a convex curve $C$ in the plane and a point $z$ in its interior, we denote by $D_1(z)$ the smallest and by $D_2(z)$ the greatest distance of the points of $C$ from $z$. Let $C$ be of class $C^2$ and suppose that its radius of curvature function $\rho$ satisfies $0 < R_1 \leq \rho \leq R_2$ with some constants $R_1$, $R_2$. If $z$ is the perimeter centroid of $C$, then Goodman [4] proved that $R_1 \leq D_1(z) \leq D_2(z) \leq R_2$. He asked if here $z$ may be replaced by Steiner's curvature centroid. In the present note we obtain a general result of this kind, valid for ovaloids in $\mathbb{R}^d$ and for the whole series of area and curvature centroids (Theorem 2). In its proof we use an improved version of a theorem of Rauch [5] (Theorem 1 below). Our main aim is to make better known the fact that assertions of this kind can be given a stronger form (and a simpler proof) if some elementary results from the theory of convex bodies are used.

2. Ovaloids with comparable second fundamental forms. By an ovaloid we understand a closed convex hypersurface $M$ in $\mathbb{R}^d$ ($d \geq 2$) of class $C^2$ and with everywhere positive curvatures. For $x \in M$, we denote by $n(x)$ the outward unit normal vector to $M$ at $x$. The tangent space $M_x$ of $M$ at $x$ is identified with the linear subspace of $\mathbb{R}^d$ orthogonal to $n(x)$. The Weingarten map $L_x : M_x \rightarrow M_x$ is defined by $L_xv = D_vn(x)$ for $v \in M_x$, where $D_v$ denotes the directional derivative operator (as in [5]). The second fundamental form of $M$ at $x$ is given by $\Pi_x(v) = (L_xv, v)$ for $v \in M_x$. For $u \in \mathbb{R}^d \setminus \{0\}$ we denote by $p(u)$ the unique point of $M$ with $n(p(u)) = u/\|u\|$. Since the Gauss map of $M$ and its inverse are of class $C^1$, the function $p$ thus defined is of class $C^1$. If we restrict this function to the unit sphere $\Omega = \{u \in \mathbb{R}^d : \|u\| = 1\}$, we have $D_\nu p(u) = (L_p(u))^{-1}u$ [5, p. 502].

The ovaloid $M$ is the boundary of a convex body $K$. Its support function is defined by

$$H(u) = \max_{y \in K} \langle y, u \rangle \quad \text{for } u \in \mathbb{R}^d,$$

which implies $H(u) = \langle p(u), u \rangle$ and hence

(2.1) \quad \text{grad } H(u) = p(u)$
(cf. [2, p. 26]). It follows that $H$ is of class $C^2$. Let $d^2 H_u$ be the second differential of $H$ at $u$, considered as a bilinear form on $\mathbb{R}^d$. For $u \in \mathbb{R}^d \setminus \{0\}$ we denote by $\pi_u$ the orthogonal projection onto the orthogonal complement of $u$.

**Lemma.** $d^2 H_u(a, b) = \|u\|^{-1} \langle (L_{p(u)})^{-1} \pi_u a, \pi_u b \rangle$ for $a, b \in \mathbb{R}^d$.

**Proof.** By homogeneity, we may assume $\|u\| = 1$. Let $(e_1, \ldots, e_d)$ be an orthonormal basis for $\mathbb{R}^d$ with $e_d = u$. We write $(y^1, \ldots, y^d)$ for the coordinates of $y \in \mathbb{R}^d$ and $H_{ij}$ for the second partial derivatives of $H$ with respect to this basis. The homogeneity relation

$$\sum_{j=1}^d H_{ij}(y) y^j = 0 \quad \text{for } i = 1, \ldots, d$$

yields $H_{id}(u) = 0$ and hence

$$d^2 H_u(a, b) = \sum_{i,j=1}^{d-1} H_{ij}(u) a^i b^j \quad \text{for } a, b \in \mathbb{R}^d.$$ 

On the other hand, $\langle (L_{p(u)})^{-1} e_i, e_j \rangle = (D_{e_i} p(u), e_j)$ is the $j$th coordinate of $D_{e_i} p(u)$ and hence by (2.1) equal to $H_{ji}(u)$, if $i, j \leq d - 1$. This gives

$$\langle (L_{p(u)})^{-1} \pi_u a, \pi_u b \rangle = \sum_{i,j=1}^{d-1} H_{ij}(u) a^i b^j,$$

which proves the lemma.

Now let $\tilde{M}$ be a second ovaloid. All objects defined for $\tilde{M}$ as above for $M$ are distinguished by a tilde.

**Theorem 1.** If $M$ and $\tilde{M}$ are ovaloids such that $\Pi_x \geq \Pi_{\tilde{x}}$ for all points $x \in M$, $\tilde{x} \in \tilde{M}$ where the outer normal vectors to $M$ and $\tilde{M}$ are the same, then there exists a convex body $L$ with $K = K + L$.

**Proof.** The assumption $\Pi_x \geq \Pi_{\tilde{x}}$, or $L_x \geq L_{\tilde{x}}$, for $n(x) = \tilde{n}(\tilde{x})$ implies $(L_{p(u)})^{-1} \leq (\tilde{L}_{p(u)})^{-1}$ [5, p. 503], hence the lemma yields $d^2 H_u \leq d^2 H_{\tilde{u}}$ for all $u \neq 0$. Thus the function $\tilde{H} - H$ has positive semidefinite Hessian on $\mathbb{R}^d \setminus \{0\}$ and is, therefore, convex. Being positively homogeneous of degree 1, it is the support function $H_L$ of a uniquely determined convex body $L$ [2, p. 26], and the equation $\tilde{H} = H + H_L$ is equivalent to $\tilde{K} = K + L$ [2, p. 29]. The proof is complete.

Under the assumptions of Theorem 1, Rauch [5] has shown that $M$ can fit inside $\tilde{M}$. This is an immediate consequence of our strengthened version: Each point $t \in L$ satisfies $M + t \subset \tilde{K}$. The theorem as formulated by Rauch also follows: Suppose that $M, \tilde{M}$ satisfy the assumptions above and are internally tangent at some point $x$ (i.e., $x \in M \cap \tilde{M}$ and $n(x) = \tilde{n}(x) = u$, say). The point $x \in \tilde{K} = K + L$ has a representation $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in L$. From $\tilde{H}(u) = \langle x, u \rangle = \langle x_1, u \rangle + \langle x_2, u \rangle \leq H(u) + H_L(u) = \tilde{H}(u)$ we conclude that $x_1 = x$ and hence $0 \in L$, which implies $M \subset K \subset \tilde{K}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
3. The position of the curvature centroids. We generalize the result of Goodman [4]. The \( r \)th curvature centroid of the ovaloid \( M \) is defined by

\[
p_r(M) = \frac{\int_M x H_{r-1}(x) \, dA(x)}{\int_M H_{r-1} \, dA}, \quad r = 1, \ldots, d,
\]

where \( H_k \) denotes the \( k \)th elementary symmetric function of the principal curvatures of \( M \) and \( dA \) is the surface area element. In particular, \( p_1 \) is the area centroid and \( p_d \) is Steiner’s curvature centroid (for more information on curvature centroids, see [6]). By \( B(z, R) \) we denote the closed ball with centre \( z \) and radius \( R > 0 \).

**Theorem 2.** Let \( M \) be an ovaloid in \( \mathbb{R}^d \) and let \( R_1, R_2 \) be constants such that

\[0 < R_1 < p < R_2\]

whenever \( p \) is a radius of curvature of \( M \). Let \( K \) be the convex body bounded by \( M \). If \( z \) is any of the curvature centroids \( p_1(M), \ldots, p_d(M) \), then \( B(z, R_1) \subseteq K \subseteq B(z, R_2) \).

**Proof.** For \( R > 0 \) let \( \Pi(R) \) denote the second fundamental form of the sphere \( R\Omega \). The assumptions of Theorem 2 imply that the second fundamental form \( \Pi \) of \( M \) satisfies \( \Pi_u(R_1) \geq \Pi_{p(u)} \geq \Pi_u(R_2) \). From Theorem 1 (which in this special case reduces to a known argument, cf. Firey [3] or Weil [7]) we deduce the existence of convex bodies \( K_1, K_2 \) satisfying

\[
\begin{align*}
K &= R_1 B + K_1, \\
R_2 B &= K + K_2,
\end{align*}
\]

where \( B \) denotes the unit ball of \( \mathbb{R}^d \). For a strictly convex body \( L \) and for \( u \in \Omega \), we now write \( p(L, u) \) for the unique point of the boundary \( \partial L \) where \( u \) is attained as outward unit normal vector. From (2.1), (3.1), (3.2) we have

\[
\begin{align*}
p(K, u) &= R_1 u + p(K_1, u), \\
R_2 u &= p(K, u) + p(K_2, u).
\end{align*}
\]

Given any positive Borel measure \( \varphi \) on \( \Omega \) which satisfies

\[
\int_{\Omega} \varphi = 1 \quad \text{and} \quad \int_{\Omega} u \, d\varphi(u) = 0,
\]

we may define the \( \varphi \)-centroid of \( L \) by

\[
c(L) = \int_{\Omega} p(L, u) \, d\varphi(u).
\]

Since \( \varphi \) is nonnegative and normalized and since \( L \) is convex, we have \( c(L) \in L \). Now (3.3) yields \( c(K) = c(K_1) \in K_1 \), hence (3.1) implies \( K \supseteq R_1 B + c(K) \), thus \( B(c(K), R_1) \subseteq K \). Similarly, (3.4) yields \( 0 = c(K) + c(K_2) \), hence \( -c(K) \in K_2 \), and (3.2) implies \( R_2 B \supseteq K - c(K) \), thus \( K \subseteq B(c(K), R_2) \). It remains to observe that the measure \( \varphi \) defined by

\[
d\varphi = \frac{S_{n-r} \, dw}{\int_M H_{r-1} \, dA},
\]

where \( S_k(w) \) is the \( k \)th elementary symmetric function of the principal radii of curvature of \( M \) at \( p(K, u) \) and \( \omega \) is spherical Lebesgue measure, satisfies (3.5) and gives \( c(K) = p_r(M) \) (cf. [6, pp. 122–123]). This completes the proof of Theorem 2.
Finally we remark that the left-hand inclusion of Theorem 2 remains true if $z$ is the centroid $p_0(K)$ (centre of mass of $K$). This follows from the fact that $p_0(K) = p_0(R_1 B + K_1)$ is a convex combination of the points $p_0(K_1), p_1(\partial K_1), \ldots, p_d(\partial K_1)$ (see [6, formula (27)]) and hence a point of $K_1$ (this remark extends an argument of Blaschke [1]).

REFERENCES