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CLOSED CONVEX HYPERSURFACES WITH CURVATURE RESTRICTIONS

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ABSTRACT. Using elementary facts from the theory of convex bodies, we obtain an improved version of J. Rauch's [**5**] inclusion theorem for ovaloids with comparable second fundamental forms and a generalization of a result of A. W. Goodman [**4**] on the position of the curvature centroids for ovaloids with restricted radii of curvature.

1. Introduction. For a convex curve C in the plane and a point z in its interior, we denote by $D_1(z)$ the smallest and by $D_2(z)$ the greatest distance of the points of C from z. Let C be of class C^2 and suppose that its radius of curvature function ρ satisfies $0 < R_1 \leq \rho \leq R_2$ with some constants R_1, R_2. If z is the perimeter centroid of C, then Goodman [**4**] proved that $R_1 \leq D_1(z) \leq D_2(z) \leq R_2$. He asked if here z may be replaced by Steiner's curvature centroid. In the present note we obtain a general result of this kind, valid for ovaloids in $\mathbb{R}^d$ and for the whole series of area and curvature centroids (Theorem 2). In its proof we use an improved version of a theorem of Rauch [**5**] (Theorem 1 below). Our main aim is to make better known the fact that assertions of this kind can be given a stronger form (and a simpler proof) if some elementary results from the theory of convex bodies are used.

2. Ovaloids with comparable second fundamental forms. By an ovaloid we understand a closed convex hypersurface M in $\mathbb{R}^d$ $(d \geq 2)$ of class C^2 and with everywhere positive curvatures. For $x \in M$, we denote by $n(x)$ the outward unit normal vector to M at x. The tangent space M_x of M at x is identified with the linear subspace of $\mathbb{R}^d$ orthogonal to $n(x)$. The Weingarten map $L_x \colon M_x \to M_x$ is defined by $L_x v = D_v n(x)$ for $v \in M_x$, where D_v denotes the directional derivative operator (as in [**5**]). The second fundamental form of M at x is given by $\mathrm{II}_x(v) = \langle L_x v, v\rangle$ for $v \in M_x$. For $u \in \mathbb{R}^d \setminus \{0\}$ we denote by $p(u)$ the unique point of M with $n(p(u)) = u/\|u\|$. Since the Gauss map of M and its inverse are of class C^1, the function p thus defined is of class C^1. If we restrict this function to the unit sphere $\Omega = \{u \in \mathbb{R}^d \colon \|u\| = 1\}$, we have $D_v p(u) = (L_{p(u)})^{-1} v$ [**5**, p. 502].

The ovaloid M is the boundary of a convex body K. Its support function is defined by

$$H(u) = \max_{y \in K} \langle y, u\rangle \quad \text{for } u \in \mathbb{R}^d,$$

which implies $H(u) = \langle p(u), u\rangle$ and hence

$$\operatorname{grad} H(u) = p(u) \tag{2.1}$$

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(cf. [**2**, p. 26]). It follows that H is of class C^2. Let d^2H_u be the second differential of H at u, considered as a bilinear form on $\mathbb{R}^d$. For $u \in \mathbb{R}^d \setminus \{0\}$ we denote by π_u the orthogonal projection onto the orthogonal complement of u.

LEMMA. *$d^2H_u(a,b) = \|u\|^{-1}\langle (L_{p(u)})^{-1}\pi_u a, \pi_u b\rangle$ for $a, b \in \mathbb{R}^d$.*

PROOF. By homogeneity, we may assume $\|u\| = 1$. Let $(e_1, \dots, e_d)$ be an orthonormal basis for $\mathbb{R}^d$ with $e_d = u$. We write $(y^1, \dots, y^d)$ for the coordinates of $y \in \mathbb{R}^d$ and H_{ij} for the second partial derivatives of H with respect to this basis. The homogeneity relation

$$\sum_{j=1}^{d} H_{ij}(y)y^j = 0 \quad \text{for } i = 1, \dots, d$$

yields $H_{id}(u) = 0$ and hence

$$d^2H_u(a,b) = \sum_{i,j=1}^{d-1} H_{ij}(u)a^i b^j \quad \text{for } a, b \in \mathbb{R}^d.$$

On the other hand, $\langle (L_{p(u)})^{-1}e_i, e_j\rangle = \langle D_{e_i}p(u), e_j\rangle$ is the jth coordinate of $D_{e_i}p(u)$ and hence by (2.1) equal to $H_{ji}(u)$, if $i, j \le d-1$. This gives

$$\langle (L_{p(u)})^{-1}\pi_u a, \pi_u b\rangle = \sum_{i,j=1}^{d-1} H_{ij}(u)a^i b^j,$$

which proves the lemma.

Now let $\tilde{M}$ be a second ovaloid. All objects defined for $\tilde{M}$ as above for M are distinguished by a tilde.

THEOREM 1. *If M and $\tilde{M}$ are ovaloids such that $\mathrm{II}_x \ge \widetilde{\mathrm{II}}_{\tilde{x}}$ for all points $x \in M$, $\tilde{x} \in \tilde{M}$ where the outer normal vectors to M and $\tilde{M}$ are the same, then there exists a convex body L with $\tilde{K} = K + L$.*

PROOF. The assumption $\mathrm{II}_x \ge \widetilde{\mathrm{II}}_{\tilde{x}}$, or $L_x \ge \tilde{L}_{\tilde{x}}$, for $n(x) = \tilde{n}(\tilde{x})$ implies $(L_{p(u)})^{-1} \le (\tilde{L}_{\tilde{p}(u)})^{-1}$ [**5**, p. 503], hence the lemma yields $d^2H_u \le d^2\tilde{H}_u$ for all $u \ne 0$. Thus the function $\tilde{H} - H$ has positive semidefinite Hessian on $\mathbb{R}^d \setminus \{0\}$ and is, therefore, convex. Being positively homogeneous of degree 1, it is the support function H_L of a uniquely determined convex body L [**2**, p. 26], and the equation $\tilde{H} = H + H_L$ is equivalent to $\tilde{K} = K + L$ [**2**, p. 29]. The proof is complete.

Under the assumptions of Theorem 1, Rauch [**5**] has shown that M can fit inside $\tilde{M}$. This is an immediate consequence of our strengthened version: Each point $t \in L$ satisfies $M + t \in \tilde{K}$. The theorem as formulated by Rauch also follows: Suppose that $M, \tilde{M}$ satisfy the assumptions above and are internally tangent at some point x (i.e., $x \in M \cap \tilde{M}$ and $n(x) = \tilde{n}(x) = u$, say). The point $x \in \tilde{K} = K + L$ has a representation $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in L$. From $\tilde{H}(u) = \langle x, u\rangle = \langle x_1, u\rangle + \langle x_2, u\rangle \le H(u) + H_L(u) = \tilde{H}(u)$ we conclude that $x_1 = x$ and hence $0 \in L$, which implies $M \subset K \subset \tilde{K}$.

3. The position of the curvature centroids. We generalize the result of Goodman [**4**]. The rth curvature centroid of the ovaloid M is defined by

$$p_r(M) = \frac{\int_M xH_{r-1}(x)\,dA(x)}{\int_M H_{r-1}\,dA}, \qquad r = 1, \ldots, d,$$

where H_k denotes the kth elementary symmetric function of the principal curvatures of M and dA is the surface area element. In particular, p_1 is the area centroid and p_d is Steiner's curvature centroid (for more information on curvature centroids, see [**6**]). By $B(z, R)$ we denote the closed ball with centre z and radius $R > 0$.

THEOREM 2. *Let M be an ovaloid in $\mathbb{R}^d$ and let R_1, R_2 be constants such that $0 < R_1 \leq \rho \leq R_2$ whenever ρ is a radius of curvature of M. Let K be the convex body bounded by M. If z is any of the curvature centroids $p_1(M), \ldots, p_d(M)$, then $B(z, R_1) \subset K \subset B(z, R_2)$.*

PROOF. For $R > 0$ let $\mathrm{II}(R)$ denote the second fundamental form of the sphere $R\Omega$. The assumptions of Theorem 2 imply that the second fundamental form II of M satisfies $\mathrm{II}_u(R_1) \geq \mathrm{II}_{p(u)} \geq \mathrm{II}_u(R_2)$. From Theorem 1 (which in this special case reduces to a known argument, cf. Firey [**3**] or Weil [**7**]) we deduce the existence of convex bodies K_1, K_2 satisfying

$$K = R_1 B + K_1, \tag{3.1}$$

$$R_2 B = K + K_2, \tag{3.2}$$

where B denotes the unit ball of $\mathbb{R}^d$. For a strictly convex body L and for $u \in \Omega$, we now write $p(L, u)$ for the unique point of the boundary ∂L where u is attained as outward unit normal vector. From (2.1), (3.1), (3.2) we have

$$p(K, u) = R_1 u + p(K_1, u), \tag{3.3}$$

$$R_2 u = p(K, u) + p(K_2, u). \tag{3.4}$$

Given any positive Borel measure φ on Ω which satisfies

$$\int_\Omega d\varphi = 1 \quad \text{and} \quad \int_\Omega u\,d\varphi(u) = 0, \tag{3.5}$$

we may define the φ-centroid of L by

$$c(L) = \int_\Omega p(L, u)\,d\varphi(u).$$

Since φ is nonnegative and normalized and since L is convex, we have $c(L) \in L$. Now (3.3) yields $c(K) = c(K_1) \in K_1$, hence (3.1) implies $K \supset R_1 B + c(K)$, thus $B(c(K), R_1) \subset K$. Similarly, (3.4) yields $0 = c(K) + c(K_2)$, hence $-c(K) \in K_2$, and (3.2) implies $R_2 B \supset K - c(K)$, thus $K \subset B(c(K), R_2)$. It remains to observe that the measure φ defined by

$$d\varphi = \frac{S_{n-r}\,d\omega}{\int_M H_{r-1}\,dA},$$

where $S_k(u)$ is the kth elementary symmetric function of the principal radii of curvature of M at $p(K, u)$ and ω is spherical Lebesgue measure, satisfies (3.5) and gives $c(K) = p_r(M)$ (cf. [**6**, pp. 122–123]). This completes the proof of Theorem 2.

Finally we remark that the left-hand inclusion of Theorem 2 remains true if z is the centroid $p_0(K)$ (centre of mass of K). This follows from the fact that $p_0(K) = p_0(R_1 B + K_1)$ is a convex combination of the points $p_0(K_1), p_1(\partial K_1), \dots, p_d(\partial K_1)$ (see [**6**, formula (27)]) and hence a point of K_1 (this remark extends an argument of Blaschke [**1**]).

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