

## A RIGIDITY THEOREM FOR QUATERNIONIC-KÄHLER MANIFOLDS

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**ABSTRACT.** Let  $(M, g)$  be a compact quaternionic-Kähler manifold of dimension  $\geq 8$  and positive scalar curvature. It is shown that  $(M, g)$  has no nontrivial deformations through quaternionic Kähler manifolds.

Let  $(M^{4k}, g)$ ,  $k \geq 2$ , be a Riemannian manifold whose holonomy group is a subgroup of  $\mathrm{Sp}(k)\mathrm{Sp}(1)$ ; we say that  $(M, g)$  is a quaternionic-Kähler manifold. Suppose moreover that  $M$  is compact and that  $g$  has scalar curvature  $R > 0$ . The only known examples are symmetric spaces, and hence quite rigid. One may generalize this fact in the following direction: any deformation of  $g$  through quaternionic-Kähler metrics is obtained by pulling back  $g$  via a family of diffeomorphisms of  $M$ . We will prove this result by twistor methods.

To begin, let us recall Salamon's theory of quaternionic twistor spaces [S]. Each quaternionic-Kähler manifold  $(M, g)$  has associated to it a complex manifold  $Z$  together with a fibering  $p: Z \rightarrow M$  whose fibers are Riemann spheres. Provided that  $R > 0$ ,  $Z$  comes equipped with a Kähler-Einstein metric  $h$  whose restriction to the orthogonal bundle  $D$  of the vertical tangent space  $\ker p_*$  is just the pull-back  $p^*g$  of  $g$ . Moreover, the distribution  $D$  is actually a complex contact structure: there exist local nonzero holomorphic 1-forms  $\theta$  whose kernel is precisely  $D$  and such that  $\theta \wedge (d\theta)^{\wedge k}$  is nowhere zero. If  $L \rightarrow Z$  is the holomorphic line bundle  $TZ/D$ , we thus have an isomorphism  $\kappa^* \cong L^{\otimes(k+1)}$  where  $\kappa \rightarrow Z$  is the canonical line bundle.

Now since  $h$  is Kähler-Einstein with positive scalar curvature,  $\kappa^*$  and  $L$  are positive line bundles, so that  $\kappa^* \otimes L$  is also positive and  $H^1(Z, \mathcal{O}(L)) = 0$  by the Kodaira vanishing theorem; the rigidity theorem will eventually follow from this observation. Notice that the positivity of  $\kappa^*$  also implies that  $Z$  is a smooth projective algebraic variety of Kodaira dimension  $-\infty$ , facts which we will not use here but which will probably be useful at some future date in classifying such quaternionic  $(M, g)$ .

We now prove the key technical lemma of this note.

**PROPOSITION.** *Let  $X$  be a compact complex contact manifold with contact distribution  $D \subset TX$ ; let  $L = TX/D$  be the contact line bundle of  $X$ , and suppose that  $H^1(X, \mathcal{O}(L)) = 0$ . Then any small complex contact deformation of  $X$  is trivial, in the following sense: if  $\pi: \hat{X} \rightarrow \mathbf{R}$  is a smooth proper map whose fibers  $X_t = \pi^{-1}(t)$  are complex contact manifolds with complex contact structure depending smoothly*

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on  $t$ , and if  $X = X_0$ , there is a neighborhood  $I$  of  $0 \in \mathbf{R}$  such that  $\pi^{-1}(I) \cong X \times I$  in a fiber-wise complex contact manner. Moreover, if  $H^1(X_t, \mathcal{O}(L_t)) = 0$  for all  $t \in \mathbf{R}$ , where  $L_t \rightarrow X_t$  is the contact line bundle analogous to  $L \rightarrow X$ , then  $\hat{X} \cong X \times \mathbf{R}$  in a fiber-wise complex contact manner.

PROOF. By the holomorphic version of Darboux’s theorem [A],  $X$  has a holomorphic atlas of charts  $\{\phi_J : U_J \rightarrow \mathbf{C}^{2k+1}\}$  via which the contact distribution becomes the orthogonal space of

$$\theta = dz^{2k+1} + \sum_{m=1}^k dz^m \wedge dz^{m+k}.$$

We will now think of the canonical projection  $TX \rightarrow TX/D$  as a twisted 1-form  $\theta_X \in \Gamma(X, \Omega^1(L))$  which becomes  $\phi_J^* \theta$  in a local trivialization of  $L$  over the open set  $U_J \subset X$ .

Let  $\mathcal{E}$  be the sheaf of holomorphic vector fields  $V$  on  $X$  having the property that  $L_V \mathcal{O}(D) \subset \mathcal{O}(D)$ ; elements of  $\mathcal{E}$  are called infinitesimal complex contact transformations. There is a natural isomorphism between  $\mathcal{E}$  and  $\mathcal{O}(L)$  given by  $V \rightarrow V \lrcorner \theta_X$ , whose inverse is obtained in local coordinates on  $U_J$  by solving the equations

$$V \lrcorner \theta = f, \quad V \lrcorner d\theta \equiv -df \pmod{\theta},$$

where  $f\theta$  is any given local section of  $L$ ; we shall denote this inverse by  $\mu : \mathcal{O}(L) \rightarrow \mathcal{E} \subset \mathcal{O}(TX)$ .

Now equip  $\hat{X}$  with charts  $\hat{\phi}_J : \hat{U}_J \rightarrow \mathbf{C}^{2k+1} \times \mathbf{R}$  which take open sets in  $X_t$  to open sets in  $\mathbf{C}^{2k+1} \times \{t\}$  in such a way as to take  $D_t$  to the orthogonal space of  $\theta$ . We may then consider the transition functions  $\phi_{JK} : \phi_K(U_J \cap U_K) \rightarrow \phi_J(U_J \cap U_K)$  given by  $\phi_{JK} = \phi_J \circ \phi_K^{-1}$  and let  $V_{JK}(t) \in \Gamma(\pi^{-1}(t) \cap U_J \cap U_K, \mathcal{O}(\mathcal{E}_t))$  be defined by

$$V_{JK}(t) = (\phi_J^{-1}(t))_* \frac{d}{dt} \phi_{JK}(t)$$

where  $\phi_{JK}(t)$  maps one subregion of  $\mathbf{C}^{2k+1}$  biholomorphically to another by  $\phi_{JK}(t)(z) = \phi_{JK}(z, t)$ ,  $t \in \mathbf{R}$ . Since  $V_{JK}$  is an infinitesimal complex contact transformation on  $U_J \cap U_K$ , we may define  $\theta_{JK} \in \Gamma(U_J \cap U_K, \mathcal{O}(L_t))$  by

$$\theta_{JK}(t) = \mu_t^{-1}(V_{JK}(t)).$$

To prove the proposition, we now proceed as in Kodaira-Spencer theory [K] and extract a finite subcover of  $X_t$  from  $\{U_J \cap \pi^{-1}(t)\}$ , viewing  $\theta_{JK}(t)$  as a Čech cocycle for  $H^1(X_t, \mathcal{O}(L_t))$ . For  $t$  in some neighborhood  $I$  of  $0$ , the hypothesis  $H^1(X, \mathcal{O}(L)) = 0$  implies that  $H^1(X_t, \mathcal{O}(L_t)) = 0$  by the upper semicontinuity of dimension, and we can choose sections  $\theta_J \in \Gamma(U_J, \mathcal{O}(L_t))$  with  $\theta_{JK}(t) = \theta_J(t) - \theta_K(t)$ . We thus have vector fields  $V_J(t) = \mu_t(\theta_J(t))$  in  $\mathcal{E}_t$  such that  $V_{JK} = V_J - V_K$ .

On  $\phi_J(U_J \cap U_K)$ , consider the ordinary differential equation

$$d\hat{Z}/dt = -\phi_{J*} V_J, \quad \hat{Z}(Z, 0) = Z,$$

where  $\hat{Z}(Z, t) \in \mathbf{C}^{2k+1}$  for  $Z \in \mathbf{C}^{2k+1}$ ,  $t \in \mathbf{R}$ , whose solutions define a biholomorphism from a neighborhood of  $\phi_J(\pi^{-1}(t) \cap U_J)$  in  $\phi_J(U_J)$  to another neighborhood of  $\phi_J(\pi^{-1}(t) \cap U_J)$ . Replacing  $(Z, t)$  with the new coordinates  $(\hat{Z}, t)$  by refining the cover, one obtains new transition functions which are independent of  $t$ . The

proposition follows by noticing that this gives rise to isomorphisms of  $\pi^{-1}(I_\alpha)$  with  $X_{t\alpha} \times I_\alpha$  for intervals  $I_\alpha$  covering  $I$ .

Notice that if  $H^1(X_t, \mathcal{O}(L_t)) = 0$  for all  $t \in \mathbf{R}$ , we may take  $I = \mathbf{R}$ . Q.E.D.

With this tool in hand, we may prove our rigidity theorem:

**THEOREM.** *Let  $M$  be a compact  $4k$ -manifold,  $k \geq 2$ , and let  $\{g_t\}$  be a family of quaternionic-Kähler metrics on  $M$  of fixed volume depending smoothly on  $t \in \mathbf{R}$ . If  $g_0$  has positive scalar curvature, there is a family of diffeomorphisms  $\{\psi_t: M \rightarrow M\}$  depending smoothly on  $t$  such that  $\psi_t^* g_t = g_0$ .*

**PROOF.** First notice that it suffices to prove the theorem under the assumption that all the  $g_t$  have positive scalar curvature. For, if not, let  $R(t)$  be the (constant) scalar curvature of  $g_t$ , and let  $I$  be the largest open interval containing 0 and on which  $R(t)$  is positive;  $R(t)$  is a continuous function of  $t$ . If the theorem is true under the additional assumption of positive scalar curvature,  $R(t)$  is constant on  $I$ , and hence positive on some bigger interval if  $I \neq \mathbf{R}$ . By the maximality of  $I$ , we must then have  $I = \mathbf{R}$ .

Let us therefore assume that the  $g_t$  have positive scalar curvature for  $t \in \mathbf{R}$ , and let  $\pi: \hat{Z} \rightarrow \mathbf{R}$  be the family whose fibers are the twistor spaces  $Z_t$  of the quaternionic-Kähler manifolds  $(M, g_t)$ . This is a family of complex contact manifolds satisfying  $H^1(Z_t, \mathcal{O}(L_t)) = 0$  for all  $t \in \mathbf{R}$ , so that, by the proposition, there is a diffeomorphism  $\hat{\psi}: Z_0 \times \mathbf{R} \rightarrow \hat{Z}$  which sends  $Z_0 \times \{t\}$  biholomorphically onto  $Z_t$  in a contact-structure preserving fashion. Let  $\hat{\psi}_t: Z_0 \rightarrow Z_t$  via  $\hat{\psi}_t(z) = \hat{\psi}(z, t)$ .

Now  $Z_t$  possesses a *real structure*, meaning an antiholomorphic map  $\sigma_t: Z_t \rightarrow Z_t$  without fixed points such that  $\sigma_t^2 = 1$  and the fibers of  $P_t: Z_t \rightarrow M$  are invariant under  $\sigma_t$ . Let us pull these real structures back to  $Z = Z_0$  to obtain real structures  $\rho_t := \hat{\psi}_t^{-1} \sigma_t \hat{\psi}_t$  on  $Z$ .

Now  $\rho_t$  sends the contact distribution to itself, and thus defines an antilinear map of  $H^0(Z, \mathcal{O}(L))$  to itself.  $H^0(Z, \mathcal{O}(L)) = \mathfrak{g}$  is the Lie algebra of the complex Lie group  $G$  of complex contact transformations of  $Z$ ;  $\rho_t$  is a real form for  $\mathfrak{g}$  leaving fixed those infinitesimal complex contact transformations arising from Killing fields on  $(M, g_t)$ . Since the isometry group of  $(M, g_t)$  is compact,  $G$  is reductive and  $[\mathbf{H}]$  any two such real forms are conjugate via elements of  $G$ . Thus for all  $t$  we can find  $\gamma_t \in G$  depending continuously on  $t$  such that  $\rho_t = \gamma_t^{-1} \rho_0 \gamma_t$  and such that  $\gamma_0 = 1$ .

Let  $\Phi_t = \hat{\psi}_t \gamma_t^{-1}$ , so that  $\sigma_0 = \Phi_t^{-1} \sigma_t \Phi_t$ . The fibers of  $Z_t$  are precisely the  $\sigma_t$ -invariant elements of a complete analytic family of compact curves, so that  $\Phi_t$  sends the fibers of  $p_0: Z_0 \rightarrow M$  to the fibers of  $p_t: Z_t \rightarrow M$ . Consequently, there are diffeomorphisms  $\psi_t: M \rightarrow M$  making the diagram

$$\begin{array}{ccc}
 Z_0 & \xrightarrow{\Phi_t} & Z_t \\
 p_0 \downarrow & & \downarrow p_t \\
 M & \xrightarrow{\psi_t} & M
 \end{array}$$

commute. Since  $\Phi_t$  is a complex contact transformation, and since the contact structure of the twistor space determines the metric up to an overall factor,  $\psi_t^* g_t = c_t g_0$  for some  $c_t > 0$ . But since  $g_t$  and  $g_0$  have the same volume,  $c_t = 1$  and  $\psi_t$  is an isometry between  $(M, g_0)$  and  $(M, g_t)$ . Q.E.D.

REMARKS. (1) The deformation theory of complex contact manifolds given in the proposition may be extended to the case when  $H^1(X, \mathcal{O}(L)) \neq 0$  but  $H^2(X, \mathcal{O}(L)) = 0$ ; in this case,  $H^1(X, \mathcal{O}(L))$  becomes the tangent space to a universal family of complex contact deformations of  $X$ .

(2) Equipped with 20/20 hindsight, one should be able to translate the above proof into terms which avoid using the twistor space. The vanishing of  $H^1(Z, \mathcal{O}(L))$  corresponds [S2] via the Penrose transform to the triviality of solutions of a linear differential equation on  $M$  which apparently are to be interpreted as “linearized quaternionic metrics modulo diffeomorphisms”. The Kodaira vanishing argument should correspond to a Bochner vanishing argument on  $M$ .

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