

COMPACT PARALLELIZABLE FOUR DIMENSIONAL SYMPLECTIC AND COMPLEX MANIFOLDS

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ABSTRACT. Examples of compact symplectic manifolds with no complex and/or Kähler structures are presented.

1. Introduction. Many examples of compact symplectic manifolds that carry no positive definite Kähler metric are now known. Here we present some compact 4-dimensional manifolds that have symplectic structures but carry no complex structures. More generally we prove

THEOREM 1.1. *Let E^4 be a principal circle bundle over E^3 , which in turn is a principal circle bundle over a torus T^2 , so that the first Betti number of E^4 satisfies $2 \leq b_1(E^4) \leq 4$. Then*

- (i) *if $b_1(E^4) = 2$ then E^4 has symplectic but no complex structures;*
- (ii) *if $b_1(E^4) = 3$ then E^4 has both symplectic and complex structures but no positive definite Kähler metrics; however E^4 carries indefinite Kähler metrics;*
- (iii) *$b_1(E^4) = 4$ if and only if E^4 is a 4-torus T^4 .*

REMARKS. (1) Apparently the manifolds that occur in part (i) of Theorem 1.1 are the first examples of compact *symplectic* manifolds with no complex structures. Van de Ven [VdV], Yau [Ya] and Brotherton [Br] have given examples of compact 4-dimensional almost complex manifolds with no complex structures. Brotherton used Massey products to prove the nonexistence of complex structures on certain parallelizable 4-dimensional manifolds.

(2) Thurston [Th] has given an example of a compact symplectic manifold with no positive definite Kähler metric. (See also [Ab, CFG, CFL, We1].) In §3 we shall see that it is covered under part (ii) of Theorem 1.1. It is interesting to note that this example already occurs in the work of Kodaira [Kod, Theorem 19]. An explicit description of the Kodaira-Thurston example as a complex manifold is given in §3.

(3) The spaces E^4 are all real parallelizable (but only T^4 is complex parallelizable in the sense of Wang [Wa]). By a blowing up procedure one can construct *nonparallelizable* symplectic manifolds with no complex structure and/or positive definite Kähler metric [Go].

(4) Most of the manifolds considered in Theorem 1.1 have explicit matrix realizations as nilmanifolds [CM]. See also [PS], where it is proved that a compact manifold is a principal torus bundle over a torus if and only if it is a 2-step nilmanifold. The paper [BG] is also relevant.

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(5) As a corollary to part (ii) of Theorem 1.1, we observe that none of the complex structures mentioned there can be calibrated (in the sense of [Gro]) by a symplectic form, since otherwise the corresponding \mathbf{E}^4 would admit a positive definite Kähler metric.

(6) For a general discussion of symplectic manifolds constructed as fiber bundles see [We2].

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2. The topology of a principal circle bundle over a principal circle bundle over a torus. The classification of principal circle bundles is well known:

THEOREM 2.1 [Kob, p. 35, Kos, p. 133]. *There is a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold \mathbf{M} and the cohomology group $H^2(\mathbf{M}, \mathbf{Z})$. Furthermore, given an integral closed 2-form Φ on \mathbf{M} , there is a principal circle bundle $\pi: \mathbf{E} \rightarrow \mathbf{M}$ with connection form η such that Φ is the curvature of η (that is, $\pi^*(\Phi) = d\eta$).*

Now let α and β be integral closed 1-forms on \mathbf{T}^2 such that α and β are everywhere linearly independent and the cohomology class $[\alpha \wedge \beta]$ generates $H^2(\mathbf{T}^2, \mathbf{Z})$. Theorem 2.1 implies that for every integer n there is a principal circle bundle $\mathbf{E}^3 \rightarrow \mathbf{T}^2$ corresponding to $n[\alpha \wedge \beta]$ and a connection form γ on \mathbf{E}^3 chosen so that the curvature of γ is $n\alpha \wedge \beta$. The real minimal model of \mathbf{E}^3 is thus

$$M(\mathbf{E}^3) = \{\alpha, \beta, \gamma \mid d\alpha = d\beta = 0, d\gamma = n\alpha \wedge \beta\}.$$

(We use the same notation for differential forms on base spaces and their pullbacks to total spaces.) Then $H^1(\mathbf{E}^3, \mathbf{R}) = \{[\alpha], [\beta]\}$ and $H^2(\mathbf{E}^3, \mathbf{R}) = \{[\alpha \wedge \gamma], [\beta \wedge \gamma]\}$ when $n \neq 0$.

If $n = 0$, \mathbf{E}^3 is a 3-torus; otherwise \mathbf{E}^3 can be realized as the compact quotient $\Gamma_n \backslash \mathbf{H}_n$ where \mathbf{H}_n is the Lie group of matrices of the form

$$\begin{pmatrix} 1 & a & -c/n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

and Γ_n is the subgroup of \mathbf{H}_n consisting of those matrices for which a, b and c are integers.

Principal circle bundles $\mathbf{E}^4 \rightarrow \mathbf{E}^3$ are classified by $H^2(\mathbf{E}^3, \mathbf{Z})$. When $n \neq 0$ the Gysin sequence yields $H^2(\mathbf{E}^3, \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_{|n|}$; thus for each pair of integers (p, q) there is a principal circle bundle corresponding to the class $p[\alpha \wedge \gamma] + q[\beta \wedge \gamma]$. Again we use Kobayashi’s Theorem 2.1 to conclude that the connection form η of $\mathbf{E}^4 \rightarrow \mathbf{E}^3$ can be chosen so that its curvature form is precisely $p\alpha \wedge \gamma + q\beta \wedge \gamma$. It follows that when $n \neq 0$ the (real) minimal models of the \mathbf{E}^4 are given as follows:

$$M(\mathbf{E}^4) = \{\alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = 0, d\gamma = n\alpha \wedge \beta, d\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma\}.$$

Clearly, $b_1(\mathbf{E}^4) = 3$ if $p = q = 0$ and $b_1(\mathbf{E}^4) = 2$ otherwise. The case $n = 0$ is similar: for each triple of integers (p, q, r) there is an \mathbf{E}^4 with

$$M(\mathbf{E}^4) = \{\alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = d\gamma = 0, d\eta = r\alpha \wedge \beta + p\alpha \wedge \gamma + q\beta \wedge \gamma\}.$$

Then $b_1(\mathbf{E}^4) = 4$ if $r = p = q = 0$ and $b_1(\mathbf{E}^4) = 3$ otherwise.

LEMMA 2.2. *The minimal model $M(\mathbf{E}^4)$ is not formal if one of n, p, q, r is different from zero.*

PROOF. It suffices to find nonzero Massey products. Suppose $n \neq 0$ (the proof when $n = 0$ is simpler). Then the cohomology classes $[\alpha \wedge \alpha]$ and $[\alpha \wedge n\beta]$ are both zero, so that the Massey product $\langle [\alpha], [\alpha], [n\beta] \rangle$ is well defined. By definition it is represented by $\alpha \wedge \gamma$. Now $[\alpha \wedge \gamma] \neq 0$ for \mathbf{E}^3 ; it is also nonzero in the cohomology of \mathbf{E}^4 except when $p \neq 0, q = 0$. But in this case the Massey product $\langle [\beta], [\beta], [n\alpha] \rangle$ is nonzero.

3. **Proof of Theorem 1.1.** First let us note that in all cases \mathbf{E}^4 has many symplectic forms. For example

$$\Omega = (a\alpha + b\beta) \wedge \gamma + (e\alpha + f\beta) \wedge \eta$$

is closed if a, b, e, f are constants such that $fp - eq = 0$, and has maximal rank if $af - be \neq 0$.

PROOF OF (i). We use [Kod, Theorem 25]: *A [complex] surface is a deformation of an algebraic surface if and only if its first Betti number is even.* Suppose \mathbf{E}^4 with $b_1(\mathbf{E}^4) = 2$ had a complex structure. Then [Kod, Theorem 25] would imply that \mathbf{E}^4 would have a positive definite Kähler metric. But now a result of [DGMS] would imply that $M(\mathbf{E}^4)$ is formal, and this is impossible by Lemma 2.2.

REMARK. It is amusing to compare an \mathbf{E}^4 with $b_1(\mathbf{E}^4) = 2$ with the Kähler manifold $\mathbf{S}^2 \times \mathbf{T}^2$. Both are parallelizable and have the same Betti numbers. But \mathbf{E}^4 has nonzero Massey products while $\mathbf{S}^2 \times \mathbf{T}^2$ does not.

PROOF OF (ii). When $b_1(\mathbf{E}^4) = 3$ and $n \neq 0$ an explicit complex structure on \mathbf{E}^4 can be constructed as follows. Let X, Y, Z, T be the parallelization dual to $\alpha, \beta, \gamma, \eta$; the only nonzero bracket is $[X, Y] = -nZ$. Now define an almost complex structure J on \mathbf{E}^4 by $JX = Y, JZ = T$. A direct calculation shows that the Nijenhuis tensor of J vanishes; consequently J is complex. A similar construction yields a complex structure on an \mathbf{E}^4 with $n = 0$.

None of these \mathbf{E}^4 can possess a positive definite Kähler metric since $b_1(\mathbf{E}^4)$ is odd. (There are also nonzero Massey products.) Nonetheless an indefinite Kähler metric ϕ for the complex structure J can be constructed as follows. Let Ω be a symplectic form which has type $(1, 1)$ with respect to J ; for example we can take $\Omega = \alpha \wedge \gamma + \beta \wedge \eta$. Then put $\phi(U, V) = \Omega(U, JV)$ for vector fields U, V on \mathbf{E}^4 .

In general suppose that Ω is Hermitian with respect to an almost complex structure J so that the metric ϕ is given by $\phi(x, y) = \Omega(x, Jy)$. For vector fields X, Y, Z we have that

$$2\nabla_X(\Omega)(Y, Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) - \phi(X, S(Y, JZ)),$$

where S denotes the Nijenhuis tensor of J [Gra, formula (4.8)]. It follows that if J is integrable and Ω is symplectic, then ϕ is Kählerian, but possibly indefinite.

REMARK. The Kodaira-Thurston example belongs to case (ii); explicitly it is $\Gamma_{-1} \backslash H_{-1} \times \mathbf{S}^1$. As a complex manifold it has the following description. For each Gaussian integer n let

$$\mathbf{G}_n = \left\{ \left(\begin{array}{ccc} 1 & \bar{z} & w/n \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) \mid z \text{ and } w \text{ are complex} \right\}.$$

G_n is a complex manifold and as a Lie group it is left holomorphic but not right holomorphic. Let Ψ_n be the subgroup of G_n consisting of all those matrices whose elements are Gaussian integers. Then $E^4 = \Psi_n \backslash G_n$ is a nilmanifold and a complex manifold (but not a complex nilmanifold). The Kodaira-Thurston example is $\Psi_1 \backslash G_1$.

The proof of (iii) is obvious.

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