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CLEAR VISIBILITY AND L_2 SETS

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ABSTRACT. Let $S \subset R^2$ be a closed connected set whose points of local nonconvexity are compact. Suppose any two points of local nonconvexity are clearly visible from a common point of S. Then S is almost starshaped and S is 2-polygonally connected. This generalizes a result of Breen.

We begin with some definitions. Let $S \subset R^2$ and let $x, y \in S$. We say x sees y via S if $[x, y] \subset S$. Let $A \subset S$ be convex. Then S is almost starshaped with respect to A if every point of S sees some point of A via S. We say S is an L_2 set provided given $x, y \in S$ there exists a polygonal path in S joining x to y consisting of at most two closed line segments. Let l be a polygonal arc joining x to y in S. Then l is called a minimum 2-path if x does not see y via S, l is a polygonal arc consisting of two closed line segments and if l' is any other 2-path joining x to y in S, the arclength of l is less than or equal to the arclength of l'. We say x is clearly visible from y if there exists a neighborhood N of x such that y sees each point of $N \cap S$ via S. Finally, int S and Q denote the interior and points of local nonconvexity of S, respectively.

M. Breen in [**1**], has shown that if S is a compact, connected set in R^2 such that any two points of Q are clearly visible from a common point of S, then S is almost starshaped. We show that her theorem holds with the weaker hypothesis that S be closed and Q be compact and show that S is an L_2 set. We prove the following generalized version of Theorem 1 of [**1**].

THEOREM 1. *Let $S \subset R^2$ be closed and connected. Suppose Q is compact and any two points of Q are clearly visible from a common point of S. Then given any point $y \in R^2$ there is a line l through y such that $l \cap S$ is convex and S is almost starshaped with respect to $l \cap S$. Furthermore, S is an L_2 set.*

PROOF. For each $(x, y) \in Q \times Q$, our hypotheses imply the existence of a point $z_{(x,y)}$ and open sets N_x and N_y such that $z_{(x,y)}$ sees $N_x \cap S$ and $N_y \cap S$ via S. Let $A = \{(N_x, N_y) | (x, y) \in Q \times Q\}$. Since $Q \times Q$ is compact we may select a finite subcover $\{(N_{x_1}, N_{y_1}), (N_{x_2}, N_{y_2}), \ldots, (N_{x_n}, N_{y_n})\}$. Let $z_i = z_{(x_i, y_i)}$. Let K be a compact convex subset of R^2 such that $(Q \cup \{z_1, z_2, \ldots, z_n\}) \subset \operatorname{int} K$. Note $K \cap S$ is a polygonally connected set since given $w, t \in K \cap S$ there exist by Lemma 1 of [**3**] points $q_w, q_t \in Q$ such that $[w, q_w] \subset K \cap S$ and $[t, q_t] \subset K \cap S$. Hence for some i we must have that $[w, q_w] \cup [q_w, z_i] \cup [z_i, q_t] \cup [q_t, t] \subset S$.

Thus $K \cap S$ is a compact, connected set whose points of local nonconvexity are exactly Q. Note $K \cap S$ satisfies the hypotheses of Theorem 1 of Breen [**1**]. Thus given any $p \in R^2$ there exists a line l_p^k such that $l_p = l_p^k \cap K \cap S$ is convex and such

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that $K \cap S$ is almost starshaped with respect to l_p. Suppose $r \in S \sim K$. We claim r must see some point of l_p. Suppose not. Lemma 1 of [**3**], coupled with the facts that $Q \subset K \cap S$ and $K \cap S$ is almost starshaped with respect to l_p, imply that there exists a 2-path joining r to some point of l_p. Let C be the set of all such 2-paths in S. Select an element l of C of smallest arc-length. Let $l = [r, s] \cup [s, v]$ with $v \in l_p$. We claim $[r, s] \cap Q \neq \varnothing$. Suppose not. Then we may select $u \in [s, v]$, $u \neq s$, such that $\operatorname{conv}([r, s] \cup [s, u]) \cap Q = \varnothing$. Tietze's theorem then implies $[r, u] \subset S$, contradicting the minimality of l. Thus $[r, s] \cap Q \neq \varnothing$. Let q be the closest point of $Q \cap [r, s]$ to r. Note $q = r$ since $r \notin K$ and we may select a point c of local convexity of S such that $c \in K \cap (r, q)$, and thus c must see some point d of l_p. We consider cases:

Case 1. $[c, d] \cap [s, v] = \varnothing$. Consider the closed polygonal region R bounded by $[c, s] \cup [s, v] \cup [v, d] \cup [c, d]$. We note that Assertion 3 of [**1**] implies that S is simply connected (the lemma cited here holds for closed sets). Thus $R \subset S$. If R is convex then c sees v via S and we contradict the minimality of l. If R is not convex, R has at most one point of local nonconvexity, m. If $m = s$, then there exists a closed line segment $l' \subset R$ with $[c, s] \subset l'$ and $l' \cap [d, v] \neq \varnothing$, contradicting the assumption that r sees no point of l_p. If $m = c$ or $m = v$ then c sees v via S and we contradict the minimality of l. If $m = d$, then there exists a closed line segment $l' \subset R$ with $[c, d] \subset l'$ and $l' \cap [s, v] \neq \varnothing$, contradicting the minimality of l.

If the lnc point m is not one of s, c, v or d we consider subcases. Let $l_1 = [c, s]$, $l_2 = [s, v]$, $l_3 = [v, d]$, and $l_4 = [c, d]$. In order for m not to be one of s, c, v, or d there must be distinct i and j, $1 \leq i, j \leq 4$, with $m \in \operatorname{relint}(l_i) \cap \operatorname{relint}(l_j)$. (Recall that we are trying to establish the claim that r sees some point of l_p, from which it would immediately follow that S is almost starshaped with respect to l_p.) There are six subcases:

1. $i = 1$, $j = 2$. Then c sees v via S contradicting the minimality of l.
2. $i = 1$, $j = 3$. Since $[d, v] \subset l_p$, so the claim is established as $m \in l_p$.
3. $i = 1$, $j = 4$. Then r sees d s via S and establishes the claim.
4. $i = 2$, $j = 3$. Then $s \in [d, v] \subset l_p$ and the claim is established as r sees s via S.
5. $i = 2$, $j = 4$. This cannot occur since we are in Case 1.
6. $i = 3$, $j = 4$. Then $c \in [d, v] \subset l_p$ and the claim is established as r sees c via S.

This completes Case 1.

Case 2. $[c, d] \cap [s, v] \neq \varnothing$. Let $b = [c, d] \cap [s, v]$. Since $c \in (r, q)$ and q is the closest point of $Q \cap [r, s]$ to r, there exists $k \in (c, b)$ such that $\operatorname{conv}([r, c] \cup [c, k]) \cap Q = \varnothing$. Tietze's theorem then implies $\operatorname{conv}([r, c] \cup [c, k]) \subset S$. Since S is simply connected, $\operatorname{conv}([c, s] \cup [s, b] \cup [c, b]) \subset S$, and thus there exists a closed line segment $l' \subset S$ with $[r, k] \subset l'$ and $l' \cap [s, v] \neq \varnothing$, contradicting the minimality of l. Thus, r must see some point of l_p and so S is almost starshaped with respect to l_p. This concludes Case 2.

REMARK 1. The previous argument has established that given any compact convex set K with $(Q \cup \{z_1, \ldots, z_n\}) \subset \operatorname{int} K$ and given $p \in K$, that there exists a line l_p^K such that S is almost starshaped with respect to the convex set $l_p^K \cap K \cap S$.

REMARK 2. If S is almost starshaped with respect to A and $A \subset B \subset S$, then S is almost starshaped with respect to B.

Let $y \in S$. Choose a compact convex set K such that $y \in K$ and such that $(Q \cup \{z_1, \ldots, z_n\}) \subset \operatorname{int} K$. It follows from Remark 1 that there exists a line l_y^K such that S is almost starshaped with respect to $l_y^K \cap K \cap S$. Let $L = l_y^K \cap S$. It follows from Remark 2 that S is almost starshaped with L. We assert that L is convex. Let $J = l_y^K \cap K \cap S$ and recall by Remark 1 that J is convex. Let $a, b \in L$, and let $c \in [a, b]$. Since L and J are collinear and S is almost starshaped with respect to the convex set $J \subset S$, we must have that $c \in (\operatorname{conv}(\{a\} \cup J)) \cup (\operatorname{conv}(\{b\} \cup J)) \subset S$. Thus $c \in L$ and L is convex. Thus letting $l = l_y^K$, we see that the first part of the theorem is established.

In order to show that S is an L_2 set, let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact convex sets such that $\bigcup_{n=1}^{\infty} K_n = R^2$, and for all n, $K_n \subset K_{n+1}$ and $K \subset K_n$. Then for each n, $K_n \cap S$ satisfies the hypotheses of Theorem 1 of Breen [**1**]. Since any two points of S are contained in the same K_n for some n, to show S is L_2 it suffices to prove that for each $K_n \cap S$ is L_2. Let $x, y \in K_n \cap S$. By Theorem 1 of [**1**], there exist closed line segments l_x and l_y such that $l_x \cup l_y \subset K_n \cap S$ and $K_n \cap S$ is almost starshaped with respect to l_x and l_y. Let M_x and M_y be maximal convex subsets of $K_n \cap S$ containing l_x and l_y respectively. The proof techniques of Theorem 2.9 of Horn and Valentine [**2**] may now be employed to show $M_x \cap M_y \neq \emptyset$. It follows that $K_n \cap S$ is an L_2 set and we are done.

Bibliography

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