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UNDEFINABLE CLASSES AND DEFINABLE ELEMENTS IN MODELS OF SET THEORY AND ARITHMETIC

ALI ENAYAT

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ABSTRACT. Every countable model $\mathbf{M}$ of PA or ZFC, by a theorem of S. Simpson, has a "class" X which has the curious property: Every element of the expanded structure $(\mathbf{M}, X)$ is definable. Here we prove:

THEOREM A. *Every completion T of* PA *has a countable model $\mathbf{M}$ (indeed there are 2^ω many such $\mathbf{M}$'s for each T) which is not pointwise definable and yet becomes pointwise definable upon adjoining any undefinable class X to $\mathbf{M}$.*

THEOREM B. *Let $\mathbf{M} \models$ ZF + "V = HOD" be a well-founded model of any cardinality. There exists an undefinable class X such that the definable points of $\mathbf{M}$ and $(\mathbf{M}, X)$ coincide.*

THEOREM C. *Let $\mathbf{M} \models$ PA or ZF + "V = HOD". There exists an undefinable class X such that the definable points of $\mathbf{M}$ and $(\mathbf{M}, X)$ coincide if one of the conditions below is satisfied.*

(A) *The definable elements of $\mathbf{M}$ are cofinal in $\mathbf{M}$.*

(B) *$\mathbf{M}$ is recursively saturated and* $\mathrm{cf}(\mathbf{M}) = \omega$.

Let $\mathbf{M}$ be a model of Peano arithmetic PA (or Zermelo-Fraenkel set theory ZF). A subset X of $\mathbf{M}$ is said to be a *class* of $\mathbf{M}$ if the expanded structure $(\mathbf{M}, X)$ continues to satisfy the induction scheme (replacement scheme) for formulas of the extended language.

S. Simpson [**Si**], employing the notion of forcing introduced by Feferman in [**F**] proved the following surprising result:

THEOREM (SIMPSON). *Let $\mathbf{M}$ be a countable model of* PA *or* ZFC. *There exists a class X such that every element of $\mathbf{M}$ is definable in $(\mathbf{M}, X)$.*

In view of this theorem we ask the question: Does every countable model of PA or ZFC have a class X such that *no new definable elements* appear in $(\mathbf{M}, X)$? Of course to make the question nontrivial, we should also stipulate that X is to be an undefinable subset of $\mathbf{M}$. The "obvious" answer of "yes" turns out to be the wrong one, as witnessed by Theorem A below.

THEOREM A. *Every completion T of* PA *has continuum—many pairwise non-isomorphic models $\mathbf{M}$ with the property: for every class X of $\mathbf{M}$, if X is not first order definable by parameters, then every element of $\mathbf{M}$ is definable in $(\mathbf{M}, X)$.*

PROOF. Let $\mathbf{M}_0$ be the atomic model of T. By Gaifman [**G**] there exist 2^ω-many pairwise nonisomorphic $\mathbf{M}$'s each of which is a *minimal conservative* elementary

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(end) extension of $\mathbf{M}_0$, i.e.,

(a) $\mathbf{M}_0 \preccurlyeq \mathbf{N} \preccurlyeq \mathbf{M} \Rightarrow (\mathbf{M}_0 = \mathbf{N}$ or $\mathbf{N} = \mathbf{M})$.

(b) For every (parameter) definable $X \subseteq \mathbf{M}$, $X \cap \mathbf{M}_0$ is (parameter) definable in $\mathbf{M}_0$.

Given an element e of $\mathbf{M}$, let $(< e)$ denote the set of predecessors of e in $\mathbf{M}$. Note that if X is a class of $\mathbf{M}$, then for each $m \in \mathbf{M}$, $X \cap (< m)$ is "coded". Therefore, $X \cap \mathbf{M}_0$ is definable by a formula $\Psi(\cdot, \vec{a})$, where $\vec{a} \in \mathbf{M}_0$, since if $b \in \mathbf{M} - \mathbf{M}_0$, $(< b) \cap X$ is a definable subset of $\mathbf{M}$ and by (b) above, its intersection with $\mathbf{M}_0$ must be definable.

Furthermore, if X is *not* definable by parameters, then the element m defined in $(\mathbf{M}, X)$, as the first x witnessing X and $\Psi(\cdot, \vec{a})$ to diverge, must be in $\mathbf{M} - \mathbf{M}_0$. But if $(\mathbf{M}, X)$ defines *one* element in $\mathbf{M} - \mathbf{M}_0$ then by the minimality of $\mathbf{M}$ and the fact that there are definable Skolem functions, it must define *every* element of $\mathbf{M}$. □

Note that the proof of Theorem A does not go through for models of set theory since by [**Ka** and **E1**] *no* model of ZFC has a conservative elementary end extension, and indeed as shown in [**E2**], conservative elementary extensions must be *cofinal*. Minimal elementary end extensions of models of set theory on the other hand are possible, at least in the presence of a definable (global) well ordering. See [**Kn**, Lemma 2.3 or **Sh**, Theorem 2.1].

We do not know whether the statement of Theorem A is true when PA is replaced by ZF or even ZF + "V = HOD". However, we have the following positive result.

THEOREM B. *Let* $\mathbf{M}$ *be a well-founded model of* ZF + "V = HOD" *of any cardinality. There exists an undefinable class* X *such that the definable elements of* $(\mathbf{M}, X)$ *and* $\mathbf{M}$ *coincide.*

PROOF. We intend to use "Feferman-forcing" in the context of set theory. The forcing conditions are functions p mapping some ordinal α into $2 = \{0, 1\}$. The *forcing language* is the first order language whose alphabet consists of the binary relation $\in$, the unary predicate G, and a constant $\mathbf{m}$ for every element $m \in \mathbf{M}$. Forcing is defined inductively as usual, and for each formula $\varphi(G, \vec{u})$, and any forcing condition p, the relation $p \Vdash \varphi(G, \vec{u})$ (between p and $\vec{u}$) is definable by some formula, $\text{Force}_\varphi(p, \vec{u})$, in the language of $\{\in\}$. We recommend [**Kn**] for more detail.

The proof falls naturally into two cases.

Case (1). The definable elements of $\mathbf{M}$ are cofinal in $\mathbf{M}$.

Case (2). Not Case (1).

Proof of Case (1). Let $A = \langle a_n : n < \omega \rangle$ be a cofinal ω-sequence of definable ordinals of $\mathbf{M}$ and let $\langle \varphi_n(G, \vec{u}), b_n \rangle_{n \in \omega}$ be an enumeration of the Cartesian product $A \times F$ where F is the set of formulas $\varphi(G, \vec{u})$ ($\vec{u}$ is the sequence of free variables of φ) in the language $\{\in, G\}$. We shall inductively construct a sequence S of forcing conditions $\langle p_n : n < \omega \rangle$ such that each p_n is a definable element of $\mathbf{M}$, and S is generic over $\mathbf{M}$.

$$p_0 = (\mu p)(\forall m \in R(b_0)(p \text{ decides } \varphi_0(G, \vec{m}))),$$

$$p_{n+1} = (\mu p \geq p_n)(\forall m \in R(b_{n+1})(p \text{ decides } \varphi_{n+1}(G, \vec{m}))).$$

Here μ is the "least" operator available since we are assuming "$V = \text{HOD}$", and "p decides φ" means $p \Vdash \varphi$ or $p \Vdash \neg\varphi$. Let $\mathbf{M}_0$ be the elementary (cofinal) submodel of $\mathbf{M}$ consisting of definable elements. It is clear that $S = \langle p_n : n < \omega\rangle$ determines a unique generic $X \subseteq \text{Ord}(\mathbf{M}_0)$, as well as $X^* \subseteq \text{Ord}(\mathbf{M})$, such that $(\mathbf{M}_0, X) \prec (\mathbf{M}, X^*)$. But if $m \in M$ is definable in $(\mathbf{M}, X^*)$ by some formula $\Psi(G, \cdot)$ then we have

$$(\mathbf{M}, X^*) \vDash (\exists! x \Psi(G, x)) \wedge \Psi(G, \mathbf{m}),$$

which implies

$$(\mathbf{M}_0, X) \vDash \Psi(G, \mathbf{n}), \quad \text{for some } n \in M_0.$$

Since $(\mathbf{M}_0, X) \prec (\mathbf{M}, X^*)$, $m = n$. Therefore all the definable elements of $(\mathbf{M}, X^*)$ lie in M_0, all members of which are definable in $\mathbf{M}_0$.

Note that we did not use the well-foundedness of M in Case (1).

Case (2). In this case the minimal elementary submodel $\mathbf{M}_0$ is not cofinal in $\mathbf{M}$ and therefore by well-foundedness, there exists an ordinal $\alpha_0 \in \mathbf{M}$ which is the *supremum* of the ordinals of $\mathbf{M}_0$. Note that, by the "Factoring Theorem":

$$\mathbf{M}_0 \prec_c (R(\alpha_0))^{\mathbf{M}} \prec_e \mathbf{M}$$

(see Chapter 25 of [**Ke**] for a proof).

Now *inside* M argue as follows: $(R(\alpha_0), \in)$ is a model of ZF + "$V = \text{HOD}$" whose definable elements form a cofinal subset of $R(\alpha_0)$, hence by an (internal) application of the proof of Case (1), there exists an $X_1 \subseteq \alpha_0$, such that X_1 is generic over $(R(\alpha_0), \in)$, and the definable elements of $(R(\alpha_0), \in)$ and $(R(\alpha_0), \in, X_1)$ coincide.

Now we exploit the fact that $X_1 \in \mathbf{M}$ to extend X_1 to a generic X over $\mathbf{M}$. The proof falls into two cases again.

Case 2(A). $\text{cf}(\mathbf{M}) = \omega$.

Case 2(B). $\text{cf}(\mathbf{M}) > \omega$.

Case 2(A). This is the easier case: construct any generic X over $\mathbf{M}$ extending X_1. This can be done by taking care of many formulas at a time as in the construction of Case (1), and we leave it to the reader. To see that $(\mathbf{M}_{\alpha_0}, X_1) \prec (\mathbf{M}, X)$, suppose $(\mathbf{M}_{\alpha_0}, X_1) \vDash \varphi(G, \vec{\mathbf{m}})$, then for some $p \in X_1$,

$$\mathbf{M}_{\alpha_0} \vDash \text{“}p \Vdash \varphi(G, \vec{\mathbf{m}})\text{”},$$

which implies

$$\mathbf{M} \vDash \text{“}p \Vdash \varphi(G, \vec{\mathbf{m}})\text{”}, \text{ since } \mathbf{M}_{\alpha_0} \prec \mathbf{M}.$$

But $p \in X$ as well, so $\mathbf{M} \vDash \varphi(G, \vec{\mathbf{m}})$, and we are done.

Case 2(B). Here we use a clever trick due to M. Yasumoto who first used it to produce undefinable classes for any well-founded model of ZF in [**Y**]. A direct consequence of the reflection theorem and the fact that $\text{cf}(\mathbf{M}) > \omega$ is that there exists a closed unbounded subset $E \subseteq \text{Ord}(\mathbf{M})$ such that for each $\alpha \in E$, the initial submodel $\mathbf{M}_\alpha = (R(\alpha))^{\mathbf{M}}$ is an *elementary submodel* of $\mathbf{M}$. Without loss of generality assume $E = \langle e_\alpha : \alpha < \eta\rangle$ where η is some ordinal, and $\mathbf{M}_{e_\alpha} = \alpha$th initial elementary submodel of $\mathbf{M}$. Our plan is to construct $G_\alpha \subseteq \text{Ord}(\mathbf{M}_{e_\alpha})$ such that

(i) $X_1 \subseteq G_\alpha$, for each $\alpha < \eta$,

(ii) G_α is generic over $\mathbf{M}_{e_\alpha}$, and $G_\alpha \in \mathbf{M}$,

(iii) whenever $\alpha < \beta < \eta$, $G_\alpha \subseteq G_\beta$.

Note that if such a sequence $\langle G_\alpha : \alpha < \eta\rangle$ is constructed, then by repeating the proof of Case 2(A), $(\mathbf{M}_0, X_1) \prec (\mathbf{M}, X)$ where $X = \bigcup_{\alpha<\eta} G_\alpha$.

To produce each G_α one argues as follows:

Suppose $\mathbf{M} \models (R(\theta) \models \text{ZF} + \text{“}V = \text{HOD”})$ (θ need *not* be in E). Then *internally* one can produce $X_\theta \in \mathbf{M}$ which is generic over $R(\theta)$, as follows:

(A) If the definable elements of $R(\theta)$ are cofinal in $R(\theta)$, then X_θ is constructed as in Case (1). Note that X_θ is absolute in the sense that the external and internal constructions outlined in Case (1) produce the same set.

(B) If the definable elements of $R(\theta)$ are not cofinal in $R(\theta)$, then $R(\theta)$ can be written as $\bigcup_{\alpha<\varsigma} R(c_\alpha)$, where ς is some ordinal, and $R(c_\alpha)$ is the αth-elementary initial submodel of $R(\beta)$. Let Y_1 be a set generic over $R(c_1)$, constructed as in (A) above (since the pointwise definable elements of $R(c_1)$ are cofinal in $R(c_1)$), and let Y_2 be the first (in the OD-ordering) generic subset of $R(c_2)$ extending Y_1. (Note that $Y_1 \in R(c_2)$ and the cofinality of $R(c_2) = \omega$.) We continue this process to get $\langle Y_\alpha : \alpha < \varsigma\rangle$ such that

$Y_{\alpha+1} = \mu Y\ (Y \supseteq Y_\alpha$ and Y is generic over $R(c_{\alpha+1}))$,

$Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$, if α is limit.

Now let $X_\theta = \bigcup_{\alpha<\varsigma} Y_\alpha$. Clearly, X_θ is generic over $R(\theta)$.

We are finally prepared to define the G_α's by $G_\alpha = X_{e_\alpha}$.

Note that conditions (i) and (ii) which we set out to satisfy are easy to verify, and condition (iii) is satisfied because of the fact that $\mathbf{M}_{e_\alpha} \prec \mathbf{M}_{e_\beta}$ whenever $\alpha < \beta < \eta$. □

THEOREM C. *If* $\mathbf{M} \models$ PA *or* ZF + “$V = \text{HOD}$” *and* $\mathbf{M}$ *satisfies condition* (I) *or* (II) *below, then there exists an undefinable class* X *such that the definable elements of* $\mathbf{M}$ *and* $(\mathbf{M}, X)$ *coincide.*

(I) $\mathbf{M}$ *is recursively saturated and* $\text{cf}(\mathbf{M}) = \omega$.

(II) *The definable elements of* $\mathbf{M}$ *are cofinal in* $\mathbf{M}$.

PROOF. (I) Let $\langle \varphi_n(G) : n < \omega\rangle$ be a recursive enumeration of the *sentences* of $\{\in, G\}$ in the case of set theory, and $\{+, \cdot, 0, 1, G\}$ in the case of arithmetic; and μ be the “least” operator available in PA and ZF + $V = \text{HOD}$.

Let us describe a recursive type $\Sigma(x) = \{\sigma_n(x) : n < \omega\}$, where

$$\sigma_0(x) \text{ says “} x \supseteq \mu p(p \text{ decides } \varphi_0(G))\text{”}$$
$$\sigma_{n+1}(x) \text{ says “} x \supseteq \mu p(p \text{ decides } \varphi_{n+1}(G)) \text{ and } \sigma_n(x)\text{”}.$$

Choose some condition $p \in M$ to realize $\Sigma(x)$ and extend p to *any* generic G over $\mathbf{M}$. By the same argument as Case 2(A) of the proof of Theorem B:

$$(\mathbf{M}_0, G \cap \mathbf{M}_0) \prec (\mathbf{M}, G),$$

where $\mathbf{M}_0$ is the minimal elementary submodel of $\mathbf{M}$. Hence the proof is complete.

(II) This is really what Case 1 of Theorem B proves. (Note that well-foundedness was not used there.) □

We close with a conjecture:

CONJECTURE. The statement of Theorem A is true with “PA” replaced by “ZF + $V = \text{HOD}$”.

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Department of Mathematics and Statistics, The American University, Washington, D.C. 20016