

UNDEFINABLE CLASSES AND DEFINABLE ELEMENTS IN MODELS OF SET THEORY AND ARITHMETIC

ALI ENAYAT

(Communicated by Thomas J. Jech)

ABSTRACT. Every countable model \mathbf{M} of PA or ZFC, by a theorem of S. Simpson, has a “class” X which has the curious property: Every element of the expanded structure (\mathbf{M}, X) is definable. Here we prove:

THEOREM A. *Every completion T of PA has a countable model \mathbf{M} (indeed there are 2^ω many such \mathbf{M} 's for each T) which is not pointwise definable and yet becomes pointwise definable upon adjoining any undefinable class X to \mathbf{M} .*

THEOREM B. *Let $\mathbf{M} \models \text{ZF} + “V = \text{HOD}”$ be a well-founded model of any cardinality. There exists an undefinable class X such that the definable points of \mathbf{M} and (\mathbf{M}, X) coincide.*

THEOREM C. *Let $\mathbf{M} \models \text{PA}$ or $\text{ZF} + “V = \text{HOD}”$. There exists an undefinable class X such that the definable points of \mathbf{M} and (\mathbf{M}, X) coincide if one of the conditions below is satisfied.*

- (A) *The definable elements of \mathbf{M} are cofinal in \mathbf{M} .*
- (B) *\mathbf{M} is recursively saturated and $\text{cf}(\mathbf{M}) = \omega$.*

Let \mathbf{M} be a model of Peano arithmetic PA (or Zermelo-Fraenkel set theory ZF). A subset X of \mathbf{M} is said to be a *class* of \mathbf{M} if the expanded structure (\mathbf{M}, X) continues to satisfy the induction scheme (replacement scheme) for formulas of the extended language.

S. Simpson [Si], employing the notion of forcing introduced by Feferman in [F] proved the following surprising result:

THEOREM (SIMPSON). *Let \mathbf{M} be a countable model of PA or ZFC. There exists a class X such that every element of \mathbf{M} is definable in (\mathbf{M}, X) .*

In view of this theorem we ask the question: Does every countable model of PA or ZFC have a class X such that *no new definable elements* appear in (\mathbf{M}, X) ? Of course to make the question nontrivial, we should also stipulate that X is to be an undefinable subset of \mathbf{M} . The “obvious” answer of “yes” turns out to be the wrong one, as witnessed by Theorem A below.

THEOREM A. *Every completion T of PA has continuum—many pairwise non-isomorphic models \mathbf{M} with the property: for every class X of \mathbf{M} , if X is not first order definable by parameters, then every element of \mathbf{M} is definable in (\mathbf{M}, X) .*

PROOF. Let \mathbf{M}_0 be the atomic model of T . By Gaifman [G] there exist 2^ω -many pairwise nonisomorphic \mathbf{M} 's each of which is a *minimal conservative* elementary

Received by the editors March 10, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 03C62, 03H15; Secondary 03C50, 03C25.

©1988 American Mathematical Society
0002-9939/88 \$1.00 + \$.25 per page

(end) extension of \mathbf{M}_0 , i.e.,

(a) $\mathbf{M}_0 \preceq \mathbf{N} \preceq \mathbf{M} \Rightarrow (\mathbf{M}_0 = \mathbf{N} \text{ or } \mathbf{N} = \mathbf{M})$.

(b) For every (parameter) definable $X \subseteq \mathbf{M}$, $X \cap \mathbf{M}_0$ is (parameter) definable in \mathbf{M}_0 .

Given an element e of \mathbf{M} , let $\langle e \rangle$ denote the set of predecessors of e in \mathbf{M} . Note that if X is a class of \mathbf{M} , then for each $m \in \mathbf{M}$, $X \cap \langle m \rangle$ is “coded”. Therefore, $X \cap \mathbf{M}_0$ is definable by a formula $\Psi(\cdot, \vec{a})$, where $\vec{a} \in \mathbf{M}_0$, since if $b \in \mathbf{M} - \mathbf{M}_0$, $\langle b \rangle \cap X$ is a definable subset of \mathbf{M} and by (b) above, its intersection with \mathbf{M}_0 must be definable.

Furthermore, if X is *not* definable by parameters, then the element m defined in (\mathbf{M}, X) , as the first x witnessing X and $\Psi(\cdot, \vec{a})$ to diverge, must be in $\mathbf{M} - \mathbf{M}_0$. But if (\mathbf{M}, X) defines *one* element in $\mathbf{M} - \mathbf{M}_0$ then by the minimality of \mathbf{M} and the fact that there are definable Skolem functions, it must define *every* element of \mathbf{M} . \square

Note that the proof of Theorem A does not go through for models of set theory since by [Ka and E1] *no* model of ZFC has a conservative elementary end extension, and indeed as shown in [E2], conservative elementary extensions must be *cofinal*. Minimal elementary end extensions of models of set theory on the other hand are possible, at least in the presence of a definable (global) well ordering. See [Kn, Lemma 2.3 or Sh, Theorem 2.1].

We do not know whether the statement of Theorem A is true when PA is replaced by ZF or even $\text{ZF} + “V = \text{HOD}”$. However, we have the following positive result.

THEOREM B. *Let \mathbf{M} be a well-founded model of $\text{ZF} + “V = \text{HOD}”$ of any cardinality. There exists an undefinable class X such that the definable elements of (\mathbf{M}, X) and \mathbf{M} coincide.*

PROOF. We intend to use “Feferman-forcing” in the context of set theory. The forcing conditions are functions p mapping some ordinal α into $2 = \{0, 1\}$. The *forcing language* is the first order language whose alphabet consists of the binary relation \in , the unary predicate G , and a constant \mathbf{m} for every element $m \in \mathbf{M}$. Forcing is defined inductively as usual, and for each formula $\varphi(G, \vec{u})$, and any forcing condition p , the relation $p \Vdash \varphi(G, \vec{u})$ (between p and \vec{u}) is definable by some formula, $\text{Force}_\varphi(p, \vec{u})$, in the language of $\{\in\}$. We recommend [Kn] for more detail.

The proof falls naturally into two cases.

Case (1). The definable elements of \mathbf{M} are cofinal in \mathbf{M} .

Case (2). Not Case (1).

Proof of Case (1). Let $A = \langle a_n : n < \omega \rangle$ be a cofinal ω -sequence of definable ordinals of \mathbf{M} and let $\langle \varphi_n(G, \vec{u}), b_n \rangle_{n \in \omega}$ be an enumeration of the Cartesian product $A \times F$ where F is the set of formulas $\varphi(G, \vec{u})$ (\vec{u} is the sequence of free variables of φ) in the language $\{\in, G\}$. We shall inductively construct a sequence S of forcing conditions $\langle p_n : n < \omega \rangle$ such that each p_n is a definable element of \mathbf{M} , and S is generic over \mathbf{M} .

$$p_0 = (\mu p)(\forall m \in R(b_0)(p \text{ decides } \varphi_0(G, \vec{m}))),$$

$$p_{n+1} = (\mu p \geq p_n)(\forall m \in R(b_{n+1})(p \text{ decides } \varphi_{n+1}(G, \vec{m}))).$$

Here μ is the “least” operator available since we are assuming “ $V = \text{HOD}$ ”, and “ p decides φ ” means $p \Vdash \varphi$ or $p \Vdash \neg\varphi$. Let \mathbf{M}_0 be the elementary (cofinal) submodel of \mathbf{M} consisting of definable elements. It is clear that $S = \langle p_n : n < \omega \rangle$ determines a unique generic $X \subseteq \text{Ord}(\mathbf{M}_0)$, as well as $X^* \subseteq \text{Ord}(\mathbf{M})$, such that $(\mathbf{M}_0, X) \prec (\mathbf{M}, X^*)$. But if $m \in M$ is definable in (\mathbf{M}, X^*) by some formula $\Psi(G, \cdot)$ then we have

$$(\mathbf{M}, X^*) \models (\exists! x \Psi(G, x)) \wedge \Psi(G, m),$$

which implies

$$(\mathbf{M}_0, X) \models \Psi(G, n), \quad \text{for some } n \in M_0.$$

Since $(\mathbf{M}_0, X) \prec (\mathbf{M}, X^*)$, $m = n$. Therefore all the definable elements of (\mathbf{M}, X^*) lie in M_0 , all members of which are definable in \mathbf{M}_0 .

Note that we did not use the well-foundedness of M in Case (1).

Case (2). In this case the minimal elementary submodel \mathbf{M}_0 is not cofinal in \mathbf{M} and therefore by well-foundedness, there exists an ordinal $\alpha_0 \in \mathbf{M}$ which is the *supremum* of the ordinals of \mathbf{M}_0 . Note that, by the “Factoring Theorem”:

$$\mathbf{M}_0 \prec_c (R(\alpha_0))^{\mathbf{M}} \prec_e \mathbf{M}$$

(see Chapter 25 of [Ke] for a proof).

Now *inside M* argue as follows: $(R(\alpha_0), \in)$ is a model of ZF + “ $V = \text{HOD}$ ” whose definable elements form a cofinal subset of $R(\alpha_0)$, hence by an (internal) application of the proof of Case (1), there exists an $X_1 \subseteq \alpha_0$, such that X_1 is generic over $(R(\alpha_0), \in)$, and the definable elements of $(R(\alpha_0), \in)$ and $(R(\alpha_0), \in, X_1)$ coincide.

Now we exploit the fact that $X_1 \in \mathbf{M}$ to extend X_1 to a generic X over \mathbf{M} . The proof falls into two cases again.

Case 2(A). $\text{cf}(\mathbf{M}) = \omega$.

Case 2(B). $\text{cf}(\mathbf{M}) > \omega$.

Case 2(A). This is the easier case: construct any generic X over \mathbf{M} extending X_1 . This can be done by taking care of many formulas at a time as in the construction of Case (1), and we leave it to the reader. To see that $(\mathbf{M}_{\alpha_0}, X_1) \prec (\mathbf{M}, X)$, suppose $(\mathbf{M}_{\alpha_0}, X_1) \models \varphi(G, \bar{m})$, then for some $p \in X_1$,

$$\mathbf{M}_{\alpha_0} \models “p \Vdash \varphi(G, \bar{m})”,$$

which implies

$$\mathbf{M} \models “p \Vdash \varphi(G, \bar{m})”, \quad \text{since } \mathbf{M}_{\alpha_0} \prec \mathbf{M}.$$

But $p \in X$ as well, so $\mathbf{M} \models \varphi(G, \bar{m})$, and we are done.

Case 2(B). Here we use a clever trick due to M. Yasumoto who first used it to produce undefinable classes for any well-founded model of ZF in [Y]. A direct consequence of the reflection theorem and the fact that $\text{cf}(\mathbf{M}) > \omega$ is that there exists a closed unbounded subset $E \subseteq \text{Ord}(\mathbf{M})$ such that for each $\alpha \in E$, the initial submodel $\mathbf{M}_\alpha = (R(\alpha))^{\mathbf{M}}$ is an *elementary submodel* of \mathbf{M} . Without loss of generality assume $E = \langle e_\alpha : \alpha < \eta \rangle$ where η is some ordinal, and $\mathbf{M}_{e_\alpha} = \alpha$ th initial elementary submodel of \mathbf{M} . Our plan is to construct $G_\alpha \subseteq \text{Ord}(\mathbf{M}_{e_\alpha})$ such that

- (i) $X_1 \subseteq G_\alpha$, for each $\alpha < \eta$,
- (ii) G_α is generic over \mathbf{M}_{e_α} , and $G_\alpha \in \mathbf{M}$,
- (iii) whenever $\alpha < \beta < \eta$, $G_\alpha \subseteq G_\beta$.

Note that if such a sequence $\langle G_\alpha : \alpha < \eta \rangle$ is constructed, then by repeating the proof of Case 2(A), $(\mathbf{M}_0, X_1) \prec (\mathbf{M}, X)$ where $X = \bigcup_{\alpha < \eta} G_\alpha$.

To produce each G_α one argues as follows:

Suppose $\mathbf{M} \models (R(\theta) \models \text{ZF} + "V = \text{HOD}")$ (θ need *not* be in E). Then *internally* one can produce $X_\theta \in \mathbf{M}$ which is generic over $R(\theta)$, as follows:

(A) If the definable elements of $R(\theta)$ are cofinal in $R(\theta)$, then X_θ is constructed as in Case (1). Note that X_θ is absolute in the sense that the external and internal constructions outlined in Case (1) produce the same set.

(B) If the definable elements of $R(\theta)$ are not cofinal in $R(\theta)$, then $R(\theta)$ can be written as $\bigcup_{\alpha < \zeta} R(c_\alpha)$, where ζ is some ordinal, and $R(c_\alpha)$ is the α th-elementary initial submodel of $R(\beta)$. Let Y_1 be a set generic over $R(c_1)$, constructed as in (A) above (since the pointwise definable elements of $R(c_1)$ are cofinal in $R(c_1)$), and let Y_2 be the first (in the OD-ordering) generic subset of $R(c_2)$ extending Y_1 . (Note that $Y_1 \in R(c_2)$ and the cofinality of $R(c_2) = \omega$.) We continue this process to get $\langle Y_\alpha : \alpha < \zeta \rangle$ such that

$$Y_{\alpha+1} = \mu Y \text{ (} Y \supseteq Y_\alpha \text{ and } Y \text{ is generic over } R(c_{\alpha+1})\text{),}$$

$$Y_\alpha = \bigcup_{\beta < \alpha} Y_\beta, \text{ if } \alpha \text{ is limit.}$$

Now let $X_\theta = \bigcup_{\alpha < \zeta} Y_\alpha$. Clearly, X_θ is generic over $R(\theta)$.

We are finally prepared to define the G_α 's by $G_\alpha = X_{e_\alpha}$.

Note that conditions (i) and (ii) which we set out to satisfy are easy to verify, and condition (iii) is satisfied because of the fact that $\mathbf{M}_{e_\alpha} < \mathbf{M}_{e_\beta}$ whenever $\alpha < \beta < \eta$. \square

THEOREM C. *If $\mathbf{M} \models \text{PA}$ or $\text{ZF} + "V = \text{HOD}"$ and \mathbf{M} satisfies condition (I) or (II) below, then there exists an undefinable class X such that the definable elements of \mathbf{M} and (\mathbf{M}, X) coincide.*

(I) \mathbf{M} is recursively saturated and $\text{cf}(\mathbf{M}) = \omega$.

(II) The definable elements of \mathbf{M} are cofinal in \mathbf{M} .

PROOF. (I) Let $\langle \varphi_n(G) : n < \omega \rangle$ be a recursive enumeration of the *sentences* of $\{\in, G\}$ in the case of set theory, and $\{+, \cdot, 0, 1, G\}$ in the case of arithmetic; and μ be the "least" operator available in PA and $\text{ZF} + V = \text{HOD}$.

Let us describe a recursive type $\Sigma(x) = \{\sigma_n(x) : n < \omega\}$, where

$$\sigma_0(x) \text{ says } "x \supseteq \mu p(p \text{ decides } \varphi_0(G))"$$

$$\sigma_{n+1}(x) \text{ says } "x \supseteq \mu p(p \text{ decides } \varphi_{n+1}(G)) \text{ and } \sigma_n(x)".$$

Choose some condition $p \in M$ to realize $\Sigma(x)$ and extend p to *any* generic G over \mathbf{M} . By the same argument as Case 2(A) of the proof of Theorem B:

$$(\mathbf{M}_0, G \cap \mathbf{M}_0) < (\mathbf{M}, G),$$

where \mathbf{M}_0 is the minimal elementary submodel of \mathbf{M} . Hence the proof is complete.

(II) This is really what Case 1 of Theorem B proves. (Note that well-foundedness was not used there.) \square

We close with a conjecture:

CONJECTURE. The statement of Theorem A is true with "PA" replaced by " $\text{ZF} + V = \text{HOD}$ ".

REFERENCES

- [E1] A. Enayat, *On certain elementary extensions of models of set theory*, Trans. Amer. Math. Soc. **283** (1984), 705–715.
- [E2] ———, *Conservative extensions of models of set theory and generalizations*, J. Symbolic Logic **51** (1986), 1005–1021.
- [F] S. Feferman, *Some applications of the notion of forcing and generic sets*, Fund. Math. **56** (1965), 324–345.
- [G] H. Gaifman, *Models and types of Peano arithmetic*, Ann. Math. Logic **9** (1976), 223–306.
- [Ka] M. Kaufmann, *Blunt and topless extensions of models of set theory*, J. Symbolic Logic **48** (1983), 1053–1073.
- [Ke] H. J. Keisler, *Model theory for infinitary logic*, North-Holland, Amsterdam, 1971.
- [Kn] J. Knight, *Hanf numbers for omitting types over particular theories*, J. Symbolic Logic **41** (1976), 583–588.
- [Sh] S. Shelah, *End extensions and number of countable models*, J. Symbolic Logic **43** (1978), 550–562.
- [Si] S. Simpson, *Forcing and models of arithmetic*, Proc. Amer. Math. Soc. **43** (1974), 193–194.
- [Y] M. Yasumoto, *Classes on models of ZF*, J. Math. Soc. Japan **32** (1980), 615–621.

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE AMERICAN UNIVERSITY,
WASHINGTON, D.C. 20016