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PARAPRODUCTS AND COMMUTATORS OF MARTINGALE TRANSFORMS

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ABSTRACT. We define paraproducts in a probabilistic setting and study their boundedness properties. As an application of paraproduct techniques, we prove a generalization of the commutator result of Coifman-Rochberg-Weiss **[6]**.

1. Introduction. Let M_k be a martingale with respect to the σ-fields $\mathscr{F}_k$ and let $d_k = M_k - M_{k-1}$. We call d_k the difference sequence of the martingale M_k. Let e_k be a sequence of random variables with e_k measurable with respect to $\mathscr{F}_{k-1}$, that is, e_k is predictable, and $e_0 = 0$. Assume $|e_k| \leq 1$ for all k. Let $V_k = \sum_{j=0}^k e_j d_j$. Then V_k is called the martingale transform of M_k by e_k. These operators were introduced by D. L. Burkholder in **[2]**. They have become a powerful tool in the interplay between probabilistic harmonic analysis and harmonic analysis in Euclidean spaces. Most recently they have received a great deal of attention because of their connections to the geometry of Banach spaces and vector valued singular integrals **[4]**. The purpose of this paper is to study a particular type of martingale transform, whose definition is motivated by the paraproducts of Bony **[1]**.

Let B_t be an n-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}_t = \sigma(B_s\colon 0 \leq s \leq t)$ and for any $X \in L^2(\Omega)$ we will use $t \mapsto X_t$ to denote the almost surely continuous version of $t \mapsto E[X|\mathscr{F}_t]$. To simplify the notation, we will just write L^p instead of $L^p(\Omega)$ where there is no danger of confusion.

DEFINITION 1.1. If $X, Y \in L^2$, we define the *paraproduct* of X and Y by

$$\mathscr{P}(X,Y) = \int_0^\infty X_t\, dY_t \tag{1.1}$$

and the *remainder* term of X and Y by

$$\mathscr{R}(X,Y) = \int_0^\infty d\langle X, Y\rangle_t = \langle X, Y\rangle \tag{1.2}$$

where $\langle X, Y\rangle_t$ is the covariance process of X and Y.

Notice that $\mathscr{P}(X,Y) \neq \mathscr{P}(Y,X)$ in general but we always have $\mathscr{R}(X,Y) = \mathscr{R}(Y,X)$. Also, it follows from the Itô formula that

$$X_t Y_t = \mathscr{P}(X,Y)_t + \mathscr{P}(Y,X)_t + \mathscr{R}(X,Y)_t. \tag{1.3}$$

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While paraproducts are originally defined for $X, Y \in L^2$, we will of course extend them to operators on L^p by continuity whenever possible. So we are led to consider the questions of boundedness of paraproducts. If $X \in L^\infty$, then it follows from the Burkholder-Gundy inequalities that $\mathscr{P}(X,\cdot)\colon L^p \to L^p$ for $1 < p < \infty$. The boundedness properties of paraproducts, however, are much richer than this. First, we recall a few definitions. The space H^p, $0 < p < \infty$, consists of those functions X such that the maximal function $\sup_t |X_t| = X^*$ belongs to L^p. Doob's inequality implies that $H^p = L^p$ for $1 < p < \infty$. The quadratic variation $\langle X\rangle_t$ is the continuous increasing process which makes $X_t^2 - \langle X\rangle_t$ a martingale. We simply write $\langle X\rangle$ for $\langle X\rangle_\infty$. The Burkholder-Gundy inequalities say that $\|X^*\|_p \approx \|\langle X\rangle^{1/2}\|_p$ for $0 < p < \infty$ (see [**7**, p. 154]). For $1 < p < \infty$, $\|X^*\|_p \approx \|X\|_p$ where $X \in L^p$ such that $X_t = E[X|\mathscr{F}_t]$. The space BMO consists of those functions X in L^2 such that for all stopping times τ,

$$E[|X - X_\tau|^2|\mathscr{F}_\tau] \le c \quad \text{a.e.}$$

for some constant c. The square root of the smallest constant for which this holds is the BMO norm of X. Often we shall use the equivalent formulations

$$E\left[\int_\tau^\infty d\langle X\rangle_s|\mathscr{F}_\tau\right] \le c \quad \text{a.e.} \qquad \text{and} \qquad E[(X - X_\tau)^2] \le cP[\tau < \infty].$$

The following results are in analogy with those for analytic paraproducts proved by Coifman and Meyer [**5**]. Notice, however, that in our case we seem to be able to prove more. Chao and Long have independently obtained similar results for discrete martingales (personal communication).

THEOREM 1.1. (i) $\mathscr{P}(\cdot,\cdot)\colon (H^p, \text{BMO}) \to H^p$, $0 < p < \infty$,
(ii) $\mathscr{P}(\cdot,\cdot)\colon (L^\infty, \text{BMO}) \to \text{BMO}$,
(iii) $\mathscr{R}(\cdot,\cdot)\colon (H^p, \text{BMO}) \to L^p$, $1 \le p \le \infty$,
(iv) $\mathscr{P}(\cdot,\cdot)\colon (H^p, H^q) \to H^r$, $0 < p, q < \infty$, $1/r = 1/p + 1/q$,
(v) $\mathscr{R}(\cdot,\cdot)\colon (H^p, H^q) \to L^r$, $0 < p, q < \infty$, $1/r = 1/p + 1/q$.

$H^1 \ne L^1$, and we have the following result.

THEOREM 1.2. (i) $\mathscr{P}(\cdot,\cdot)\colon (L^1, \text{BMO}) \to WL^1$,
(ii) $\mathscr{R}(\cdot,\cdot)\colon (L^1, \text{BMO}) \to WL^1$. *Here WL^1 means weak L^1.*

For any $X \in L^2$, there is an essentially unique $\mathscr{F}_t$-adapted $\mathbf{R}^n$-valued map $s \mapsto H_s$ such that $E[\int_0^\infty |H_s|^2\, ds] < \infty$ and $X_t = E[X] + \int_0^t H_s \cdot dB_s$ (a.s. P). We now have the following.

DEFINITION 1.2. Let $A\colon [0,\infty) \to \mathbf{R}^n \otimes \mathbf{R}^n$ be adapted to $\mathscr{F}_t$ and bounded. The *martingale transform* of $X \in L^2$ by A is

$$(A * X) \equiv \int_0^\infty A_s H_s \cdot dB_s.$$

It should be noted that $(A * X)_t = \int_0^t A_s H_s \cdot dB_s$.

THEOREM 1.3. *Let $X \in L^p$, $Y \in \text{BMO}$ and $A\colon [0,\infty) \to \mathbf{R}^n \otimes \mathbf{R}^n$ be adapted to $\mathscr{F}_t$ and bounded. We define the commutator $C_Y^A(X) = Y(A * X) - A * (XY)$. Then $C_Y^A \in L^p$ for $1 < p < \infty$ and*

$$\|C_Y^A(X)\|_p \le C_p \|A\|_\infty \|Y\|_{\text{BMO}} \|X\|_p.$$

Theorems 1.1 and 1.2 are proved in §2. Here we also discuss some other properties of $\mathscr{P}(X,Y)$ and $\mathscr{R}(X,Y)$. Theorem 1.3 is proved in §3. From this we deduce the basic result of R. Coifman, R. Rochberg and G. Weiss [**6**] that the commutator of the conjugate operator in the circle and multiplication by a BMO function is bounded in L^p, $1 < p < \infty$. A similar result follows for Riesz transforms in $\mathbf{R}^n$ and multiplier operators of Laplace transform type. That is, our result holds for all those operators that can be obtained as conditional expectations of martingale transforms.

2. Boundedness of paraproducts and remainders. In this section, we prove Theorems 1.1 and 1.2, as well as some other results. We start by computing the adjoints of our operators.

THEOREM 2.1. *Let $X, Y, Z \in L^\infty$. Then*
(i) $E[\mathscr{P}(X,Y)Z] = E[Y\mathscr{P}(X,Z)]$,
(ii) $E[\mathscr{P}(X,Y)Z] = E[X\mathscr{R}(Y,Z)]$.

PROOF.

$$\begin{aligned} E[\mathscr{P}(X,Y)Z] &= E\left[\left(\int_0^\infty X_t\,dY_t\right)Z\right] = E\left[\int_0^\infty X_t\,d\langle Y,Z\rangle_t\right] \\ &= E\left[Y\left(\int_0^\infty X_t\,dZ_t\right)\right] = E[Y\mathscr{P}(X,Z)]. \end{aligned}$$

This proves (i). To prove (ii), we start the same way but now we write $\langle Y,Z\rangle_t$ as the difference of a pair of nondecreasing processes of finite variation, $\langle Y,Z\rangle_t = A_t - B_t$. A_t and B_t are the processes of positive and negative variation respectively. By stopping, we can assume A_t and B_t are bounded. Then (see [**7**, p. 191])

$$E\left[\int_0^\infty X_t\,dA_t\right] = E\left[\int_0^\infty E[X|\mathscr{F}_t]\,dA_t\right] = E\left[\int_0^\infty X\,dA_t\right] = E[XA_\infty].$$

So

$$\begin{aligned} E[\mathscr{P}(X,Y)Z] &= E\left[\int_0^\infty X_t\,d\langle Y,Z\rangle_t\right] = E\left[\int_0^\infty X_t\,dA_t\right] - E\left[\int_0^\infty X_t\,dB_t\right] \\ &= E[XA_\infty] - E[XB_\infty] = E[X(A_\infty - B_\infty)] \\ &= E[X\langle Y,Z\rangle] = E[X\mathscr{R}(Y,Z)] \end{aligned}$$

which proves (ii).

PROOF OF THEOREM 1.1. We start by proving part (iii). First, we recall the following lemma.

LEMMA 2.1 (GARSIA [**9**]). *Let A_t be a positive continuous increasing process with $A_0 = 0$. If there is a positive random variable Y such that $E[A_\infty - A_t|\mathscr{F}_t] \le E[Y|\mathscr{F}_t]$ for any time t, then, for $1 \le p < \infty$, $E[(A_\infty)^p] \le p^pE[Y^p]$.*

We may assume, by stopping, that X is bounded. Write the process of finite variation $\langle X,Y\rangle_t = A_t - B_t$ where A_t and B_t are nonnegative nondecreasing processes. A_t and B_t are the positive and negative variations of $\langle X,Y\rangle_t$ respectively. Then

$$\begin{aligned} E[((A_\infty + B_\infty) - (A_t + B_t))|\mathscr{F}_t] &= E\left[\int_t^\infty |d\langle X,Y\rangle_s|\,\middle|\,\mathscr{F}_t\right] \\ &\le c\|Y\|_{\mathrm{BMO}}E[\langle X\rangle^{1/2}|\mathscr{F}_t] \end{aligned}$$

where we have applied the local version of Fefferman's inequality [**7**, p. 190]. Since this holds for any time t, we may apply Lemma 2.1 to conclude that

$$E[(A+B)^p] \le c_p \|Y\|_{\mathrm{BMO}}^p E[\langle X\rangle^{p/2}] \le c_p c_r \|Y\|_{\mathrm{BMO}}^p E[|X|^p]$$

since $1 \le p < \infty$. This completes the proof of part (iii).

We are now ready to prove part (i). For $1 < p < \infty$, (i) follows from (ii) in Theorem 2.1 together with (iii) in Theorem 1.1.

For the case $0 < p \le 1$ of (i), we shall use the atomic decomposition. A function $A \in H^p$ is an atom if there is a stopping time τ such that

$$A_t \equiv 0 \quad \text{for } t \le \tau \tag{2.1}$$

and

$$(A^*)^p \le P[\tau < \infty]^{-1}. \tag{2.2}$$

LEMMA 2.2. *For all $X \in H^p$, $0 < p \le 1$, there exists a sequence of atoms $A^n, n \in \mathbf{Z}$ and a sequence of numbers $b_n, n \in \mathbf{Z}$, such that*

$$X_t = \sum_{n=-\infty}^{\infty} b_n A_t^n \tag{2.3}$$

and

$$c_p E[(X^*)^p] \le \sum_{n=-\infty}^{\infty} |b_n|^p \le C_p E[(X^*)^p]. \tag{2.4}$$

The proof of this lemma may be found in [**7**, p. 216].

LEMMA 2.3. *If $Y \in \mathrm{BMO}$ then for any stopping time τ and any $0 < p \le 1$*

$$E[(\langle Y\rangle_\infty - \langle Y\rangle_\tau)^{p/2}] \le \|Y\|_{\mathrm{BMO}}^p P[\tau < \infty]. \tag{2.5}$$

PROOF. By Hölder's inequality, for $0 < p \le 2$,

$$E[(\langle Y\rangle_\infty - \langle Y\rangle_\tau)^{p/2}] \le E[\langle Y\rangle_\infty - \langle Y\rangle_\tau]^{p/2} P[\tau < \infty]^{1-p/2} \le \|Y\|_{\mathrm{BMO}}^p P[\tau < \infty].$$

Now suppose A is an atom in H^p, $0 < p \le 1$. Then

$$\begin{aligned}
\|\mathscr{P}(A,Y)\|_{H^p}^p = E[\langle \mathscr{P}(A,Y)\rangle^{p/2}] &= E\left[\left(\int_\tau^\infty A_s^2\, d\langle Y\rangle_s\right)^{p/2}\right] \\
&\le E\left[(A^*)^p \left(\int_\tau^\infty d\langle Y\rangle_s\right)^{p/2}\right] \\
&\le P[\tau < \infty]^{-1} E[(\langle Y\rangle_\infty - \langle Y\rangle_\tau)^{p/2}] \le C_p \|Y\|_{\mathrm{BMO}}^p
\end{aligned}$$

where the last inequality follows from Lemma 2.3. For general $X \in H^p$ the result now follows from (2.3), (2.4) and the above. Part (i) of Theorem 1.1 is now proved for all $0 < p < \infty$.

It remains to prove parts (ii), (iv) and (v). We start with (ii). Let τ be any stopping time

$$\begin{aligned}
E[|\mathscr{P}(X,Y)_\infty - \mathscr{P}(X,Y)_\tau|^2 | \mathscr{F}_\tau] &= E\left[\int_\tau^\infty X_s^2\, d\langle Y\rangle_s | \mathscr{F}_\tau\right] \\
&\le \|X\|_\infty^2 E[\langle Y\rangle - \langle Y\rangle_\tau | \mathscr{F}_\tau] \\
&= \|X\|_\infty^2 E[|Y - Y_\tau|^2 | \mathscr{F}_\tau] \le \|X\|_\infty^2 \|Y\|_{\mathrm{BMO}}^2
\end{aligned}$$

which proves (ii).

The proof of (iv) is also very easy. Applying the Burkholder-Gundy we get

$$\begin{aligned}(E[\langle\mathscr{P}(X,Y)\rangle^{r/2}])^{1/r} &= \left(E\left[\left(\int_0^\infty X_s^2\,d\langle Y\rangle_s\right)^{r/2}\right]\right)^{1/r}\\ &\le (E[(X^*)^r\langle Y\rangle^{r/2}])^{1/r} \le E[(X^*)^p]^{1/p}E[\langle Y\rangle^{q/2}]^{1/q}\\ &\approx \|X\|_{H^p}\|Y\|_{H^q}\end{aligned}$$

where the last inequality follows from Hölder's inequality.

It remains to prove (v). Applying the Kunita-Watanabe inequality we get

$$E[|\langle X,Y\rangle|^r]^{1/r} \le (E[\langle X\rangle^{r/2}\langle Y\rangle^{r/2}])^{1/r} \le \|\langle X\rangle^{1/2}\|_p\|\langle Y\rangle^{1/2}\|_q \approx \|X\|_{H^p}\|Y\|_{H^q}$$

which proves (v). So Theorem 1.1 is completely proved.

PROOF OF THEOREM 1.2. We start with part (i). Let $X \in L^1$ and $Y \in \mathrm{BMO}$ with $\|Y\|_{\mathrm{BMO}} \le 1$. We want to show that

$$P[|\mathscr{P}(X,Y)| > \lambda] \le c\|X\|_1/\lambda. \tag{2.6}$$

We follow the argument in Burkholder [**3**] very closely. First, our proof above shows that for any stopping time τ and $1 < p < \infty$,

$$\|\mathscr{P}(X,Y)_\tau\|_p \le c_p\|Y\|_{\mathrm{BMO}}\|X_\tau\|_p. \tag{2.7}$$

Let $\lambda > 0$ and set $\tau_\lambda = \inf\{t\colon |X_t| > \lambda\}$. Then, by (i) in Theorem 1.1

$$\begin{aligned}P[|\mathscr{P}(X,Y)| > \lambda, X^* \le \lambda] &\le P\left[\left|\int_0^{\tau_\lambda} X_s\,dY_s\right| > \lambda\right] \le \frac{1}{\lambda^2}E\left[\left(\int_0^{\tau_\lambda} X_s\,dY_s\right)^2\right]\\ &= \frac{1}{\lambda^2}E[|\mathscr{P}(X,Y)_{\tau_\lambda}|^2] \le \frac{c}{\lambda^2}\|Y\|_{\mathrm{BMO}}^2\|X_{\tau_\lambda}\|_2^2\\ &\le \frac{c}{\lambda}\|Y\|_{\mathrm{BMO}}^2\|X\|_1.\end{aligned}$$

Then, applying Doob's weak type inequality we have

$$\begin{aligned}P[|\mathscr{P}(X,Y)| > \lambda] &\le P[|\mathscr{P}(X,Y) > \lambda, X^* \le \lambda] + P[X^* > \lambda]\\ &\le \frac{(1 + c\|Y\|_{\mathrm{BMO}}^2)}{\lambda}\|X\|_1\end{aligned}$$

which is (2.6) and part (i) of Theorem 1.2 is proven. Part (ii) is proved in the same way using the fact that $\|\langle X,Y\rangle_\tau\|_2 \le c\|Y\|_{\mathrm{BMO}}\|X_\tau\|_2$.

We shall now investigate some converses. Our first result is

THEOREM 2.2. *Suppose* $Y \in L^2$ *and* $\mathscr{P}(\cdot,Y)\colon L^2 \to L^2$ *with* $\|\mathscr{P}(X,Y)\|_2 \le c_2\|X\|_2$. *Then* $Y \in \mathrm{BMO}$ *and* $\|Y\|_{\mathrm{BMO}} \le c_2$.

PROOF. Let τ be any stopping time and let $\tau' = \inf\{t > \tau\colon |Y_t - Y_\tau| > \varepsilon\}$ for some fixed $\varepsilon > 0$. Then $X_t = Y_{t\wedge\tau'} - Y_{t\wedge\tau}$ is bounded and thus in L^2. Since $\mathscr{P}(\cdot,Y)$ is bounded on L^2 we have

$$E\left[\left(\int_0^\infty (Y_{t\wedge\tau'} - Y_{t\wedge\tau})\,dY_t\right)^2\right] \le c_2^2E[(Y_{\tau'} - Y_\tau)^2]. \tag{2.8}$$

But we can evaluate the right-hand side of (2.8) and see

$$\begin{aligned} E\left[\left(\int_0^\infty (Y_{t\wedge\tau'} - Y_{t\wedge\tau})\, dY_t\right)^2\right] &= E\left[\int_\tau^\infty (Y_{t\wedge\tau'} - Y_{t\wedge\tau})^2\, d\langle Y\rangle_t\right] \\ &= E\left[\int_{\tau'}^\infty \varepsilon^2\, d\langle Y\rangle_t + \int_\tau^{\tau'} (Y_t - Y_\tau)^2\, d\langle Y\rangle_t\right] \\ &\geq E\left[\int_{\tau'}^\infty \varepsilon^2\, d\langle Y\rangle_t\right]. \end{aligned}$$

So we have

$$E\left[\left(\int_0^\infty (Y_{t\wedge\tau'} - Y_{t\wedge\tau})\, dY_t\right)^2\right] \geq \varepsilon^2 E[\langle Y\rangle_\infty - \langle Y\rangle_{\tau'}]. \tag{2.9}$$

Then since $|Y_{t\wedge\tau'} - Y_{t\wedge\tau}| \leq \varepsilon$, we have

$$E[(Y_{\tau'} - Y_\tau)^2] \leq \varepsilon^2 P[\tau < \infty]. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) we see

$$E[\langle Y\rangle_\infty - \langle Y\rangle_{\tau'}] \leq c_2^2 P[\tau < \infty]. \tag{2.11}$$

Then

$$\begin{aligned} E[(Y_\infty - Y_\tau)^2] = E[\langle Y\rangle_\infty - \langle Y\rangle_\tau] &= E[\langle Y\rangle_\infty - \langle Y\rangle_{\tau'}] + E[\langle Y\rangle_{\tau'} - \langle Y\rangle_\tau] \\ &= E[\langle Y\rangle_\infty - \langle Y\rangle_{\tau'}] + E[(Y_{\tau'} - Y_\tau)^2] \leq (c_2^2 + \varepsilon^2) P[\tau < \infty]. \end{aligned}$$

So $Y \in \mathrm{BMO}$ and by letting $\varepsilon \to 0$, we see $\|Y\|_{\mathrm{BMO}} \leq c_2$.

THEOREM 2.3. *Suppose $Y \in L^2$ and $\mathscr{R}(\cdot, Y)\colon L^2 \to L^2$ with $\|\mathscr{R}(X,Y)\|_2 \leq c_2\|X\|_2$. Then $Y \in \mathrm{BMO}$ and $\|Y\|_{\mathrm{BMO}} \leq c_2^2 + 1$.*

PROOF. Let τ be any stopping time and let $X_t = Y_t - Y_{t\wedge\tau}$. Then X_t is an L^2 martingale and thus

$$\|\mathscr{R}(X,Y)\|_2 \leq c_2\|X\|_2$$

or equivalently

$$E[(\langle Y\rangle - \langle Y\rangle_\tau)^2] \leq c_2^2 E[\langle Y\rangle - \langle Y\rangle_\tau]$$

which we can write as

$$0 \leq E[c_2^2(\langle Y\rangle - \langle Y\rangle_\tau) - (\langle Y\rangle - \langle Y\rangle_\tau)^2]. \tag{2.12}$$

If k is any positive constant, we may rewrite (2.12) as

$$\begin{aligned} &E[(\langle Y\rangle - \langle Y\rangle_\tau)^2 - c_2^2(\langle Y\rangle - \langle Y\rangle_\tau); \langle Y\rangle - \langle Y\rangle_\tau \geq k] \\ &\quad\leq E[c_2^2(\langle Y\rangle - \langle Y\rangle_\tau) - (\langle Y\rangle - \langle Y\rangle_\tau)^2; \langle Y\rangle - \langle Y\rangle_\tau < k]. \end{aligned} \tag{2.13}$$

Now consider the function $f(x) = x^2 - c_2^2 x - x$. Choose k such that for all $x \geq k, f(x) > 0$, or equivalently, for all $x \geq k, x^2 - c_2^2 x \geq x$. Using this k in (2.13) we obtain

$$\begin{aligned} &E[\langle Y\rangle - \langle Y\rangle_\tau; \langle Y\rangle - \langle Y\rangle_\tau \geq k] \\ &\quad\leq E[(\langle Y\rangle - \langle Y\rangle_\tau)^2 - c_2^2(\langle Y\rangle - \langle Y\rangle_\tau); \langle Y\rangle - \langle Y\rangle_\tau \geq k] \\ &\quad\leq E[c_2^2(\langle Y\rangle - \langle Y\rangle_\tau) - (\langle Y\rangle - \langle Y\rangle_\tau)^2; \langle Y\rangle - \langle Y\rangle_\tau < k] \\ &\quad\leq c_2^2 E[\langle Y\rangle - \langle Y\rangle_\tau; \langle Y\rangle - \langle Y\rangle_\tau < k]. \end{aligned}$$

So

$$\begin{aligned} E[\langle Y\rangle - \langle Y\rangle_\tau] &= E[\langle Y\rangle - \langle Y\rangle_\tau; \langle Y\rangle - \langle Y\rangle_\tau < k] \\ &\quad + E[\langle Y\rangle - \langle Y\rangle_\tau; \langle Y\rangle - \langle Y\rangle_\tau \geq k] \\ &\leq (1+c_2^2)E[\langle Y\rangle - \langle Y\rangle_\tau; \langle Y\rangle - \langle Y\rangle_\tau < k] \\ &\leq k(1+c_2^2)P[\tau < \infty], \end{aligned}$$

and $Y \in \mathrm{BMO}$ with $\|Y\|_{\mathrm{BMO}} \leq \sqrt{k(1+c_2^2)}$. Since the smallest k that satisfies the condition above is $k = c_2^2 + 1$, we obtain $\|Y\|_{\mathrm{BMO}} \leq c_2^2 + 1$.

In both our converses, we have assumed that $Y \in L^2$. This is a weak condition compared to our conclusion that $Y \in \mathrm{BMO}$. We could weaken our initial hypothesis on Y further if we strengthen our assumptions about the boundedness of the paraproduct and the remainder term. For instance, if we assume that $\mathscr{R}(\cdot, Y)$ is bounded $L^p \to L^p$ for $1 < p < \infty$, we can show that $Y \in \bigcup_{p>1} L^p$ implies $Y \in \mathrm{BMO}$. This can be easily seen from the fact that if $Y \in L^p$ and $\|\mathscr{R}(X,Y)\|_p \leq c\|X\|_p$ then

$$\|Y\|_{2p} = E[\langle Y\rangle^p] = E[|\mathscr{R}(Y,Y)|^p] \leq c\|Y\|_p$$

so Y is in L^{2p} and $L^2 \subset L^{2p}$ since $p > 1$.

3. Boundedness of commutators of martingale transforms and projections. In this section we prove Theorem 1.3 and use it to prove the commutator result of Coifman, Rochberg and Weiss [**6**]. We have already done most of the work in §2. Let us remind the reader that if $X \in L^p$, then $A * X \in L^p$ with $\|A * X\|_p \leq c_p \|A\|_\infty \|X\|_p$ with $1 < p < \infty$.

PROOF OF THEOREM 1.3. Expanding by the Itô formula, we can write,

$$Y(A*X) = \mathscr{P}(Y, A*X) + \mathscr{P}(A*X, Y) + \mathscr{R}(Y, A*X) \tag{3.1}$$

and

$$XY = \mathscr{P}(X,Y) + \mathscr{P}(Y,X) + \mathscr{R}(X,Y). \tag{3.2}$$

Since the martingale transforms map L^p into L^p and $Y \in \mathrm{BMO}$, the results proven in §2 show that $\mathscr{P}(A*X,Y)$, $\mathscr{R}(Y,A*X)$, $A*\mathscr{P}(X,Y)$ and $A*\mathscr{R}(X,Y)$ are all in L^p with norms bounded by $c_p\|A\|_\infty \|Y\|_{\mathrm{BMO}} \|X\|_p$. To show that C_Y^A also has this property, all we have to do is show that the same holds for $\mathscr{P}(Y, A*X) - A*\mathscr{P}(Y,X)$. But a trivial calculation shows this term is actually zero and the theorem is proven.

Using the techniques of Varopoulos [**11**] to reduce matters to honest probability spaces, one can easily verify that all the results above hold for martingales on the infinite probability space of background radiation and obtain commutator results for those operators in $\mathbf{R}^n$ that can be obtained as the conditional expectation (projection) of martingale transforms. For example, we can obtain results for the Riesz transforms [**10**] and multipliers of Laplace transform type [**11**]. To keep things at an elementary level, however, we only do this for the unit disk. Let $D = \{z \in \mathbf{C}: |z| < 1\}$ and $T = \partial D$, the unit circle, equipped with the probability measure $dm = d\theta/2\pi$. Let $f \in \mathrm{BMO}(T)$ and denote by u_f its harmonic extension to D. It is well known that $f \in \mathrm{BMO}(T)$ if and only if (see [**8**, p. 238])

$$\|f\|_{\mathrm{BMO}(T)} = \left(\sup_{z\in D} \int_T |f(e^{i\theta}) - u_f(z)|^2 P_z(\theta)\, dm(\theta)\right)^{1/2} < \infty \tag{3.3}$$

where $P_z(\theta)$ is the Poisson kernel for D. We take (3.3) as our definition of BMO(T). We refer the reader to [8] for many of the important properties of BMO(T).

Let B_t be Brownian motion in D and let τ_D be its first exit time. We may write (3.3) as

$$\|f\|_{\mathrm{BMO}(T)} = \left(\sup_{z\in D} E_z[|f(B_{\tau_D}) - u_f(z)|^2]\right)^{1/2} < \infty \tag{3.4}$$

(E_z denotes expectation with $B_0 = z$). By the Itô formula and the strong Markov property, this is equivalent to

$$\|f\|_{\mathrm{BMO}(T)} = \left\| \sup_{\eta<\tau_D} \left(E\left[\int_\eta^{\tau_D} |\nabla u(B_s)|^2 \, ds | \mathscr{F}_\eta \right] \right)^{1/2} \right\|_\infty \tag{3.5}$$

where η is a stopping time and $E = E_0$. Thus, the random variable $Mf = f(B_{\tau_D})$ is in BMO. To avoid confusion we shall write BMO(Ω) for the martingale BMO and $L^p(\Omega)$ for the martingale L^p-spaces.

Now suppose $g \in L^p(T)$ for $1 < p < \infty$. If v_g is the harmonic extension of g and $A(z)$ is a 2×2 matrix valued function in D, we define the operators

$$A * (Mg) = \int_0^{\tau_D} A(B_s)\nabla v_g(B_s) \cdot dB_s$$

and

$$T_A g(e^{i\theta}) = E^\theta \left[\int_0^{\tau_D} A(B_s)\nabla v_g(B_s) \cdot dB_s \right]$$

where E^θ is the expectation with respect to Brownian motion conditions to exit D at $e^{i\theta}$. We call T_A the projection of the martingale transform $A * (Mg)$.

THEOREM 3.1. *Set $[T_A, f](g) = fT_A g - T_A(fg)$. Then for all $1 < p < \infty$*

$$\|[T_A, f](g)\|_p \le c_p \|A\| \|f\|_{\mathrm{BMO}(T)} \|g\|_p$$

where

$$\|A\| = \sup_{z\in D} \left(\sup_{\substack{v\in\mathbf{R}^2 \\ |v|=1}} |A(z)v| \right) < \infty.$$

PROOF. The key to the proof is the following claim:

$$[T_A, f](g)(e^{i\theta}) = E^\theta[C^A_{Mf}(Mg)]. \tag{3.6}$$

Once we establish (3.6), Theorem 1.3 and the fact that condition expectation contracts L^p norms (Jensen's inequality) gives

$$\begin{aligned} \|[T_A, f](g)\|_{L^p(T)} &\le \|C^A_{Mf}(Mg)\|_{L^p(\Omega)} \le c_p \|A\| \|Mf\|_{\mathrm{BMO}(\Omega)} \|Mg\|_{L^p(\Omega)} \\ &= c_p \|A\| \|f\|_{\mathrm{BMO}(T)} \|g\|_{L^p(T)}. \end{aligned}$$

To establish (3.6) we write

$$\begin{aligned} [T_A, f](g)(e^{i\theta}) &= f(\theta) E^\theta \left[\int_0^{\tau_D} A(B_s)\nabla v_g(B_s) \cdot dB_s \right] \\ &\quad - E^\theta \left[\int_0^{\tau_D} A(B_s)\nabla w_{fg}(B_s) \cdot dB_s \right] \end{aligned} \tag{3.7}$$

where w_{fg} is the harmonic extension of fg and

$$E^\theta[C^A_{Mf}(Mg)] = E^\theta[Mf(A * Mg) - A * (MfMg)]. \tag{3.8}$$

Now

$$\begin{aligned} f(\theta)E^\theta &\left[\int_0^{\tau_D} A(B_s)\nabla v_g(B_s)\cdot dB_s\right] \\ &= E^\theta\left[f(B_{\tau_D})\int_0^{\tau_D} A(B_s)\nabla v_g(B_s)\cdot dB_s\right] = E^\theta[Mf(A * Mg)]. \end{aligned}$$

So all we need prove is that

$$E^\theta\left[\int_0^{\tau_D} A(B_s)\nabla w_{fg}(B_s)\cdot dB_s\right] = E^\theta[A * (MfMg)].$$

But this is trivial since $MfMg = Mfg$.

The case when $A(z) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ gives the conjugate operator and hence the Coifman-Rochberg-Weiss result [**6**] is contained in Theorem 3.1. Many other multipliers can be obtained by taking $A(z)$ with $\|A\| < \infty$ and which commute with rotations of the plane.

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