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OCCUPATION TIME AND THE LEBESGUE MEASURE OF THE RANGE FOR A LEVY PROCESS

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ABSTRACT. We consider a Levy process on the line that is transient and with nonpolar one point sets. For $a > 0$ let $N(a)$ be the total occupation time of $[0, a]$ and $R(a)$ the Lebesgue measure of the range of the process intersected with $[0, a]$. Whenever $[0, \infty)$ is a recurrent set we show $N(a)/EN(a) - R(a)/ER(a)$ converges in the mean square to 0 as $a \to \infty$. This in turn is used to derive limit laws for $R(a)/ER(a)$ from those for $N(a)/EN(a)$.

1. Introduction. Throughout this paper X_t will be a transient Levy process that is not a compound Poisson process starting at 0. Let $T_x = \inf\{t > 0 \colon X_t = x\}$ be the hitting time of the singleton $\{x\}$. We will assume throughout that $\{0\}$ is not essentially polar, i.e. that $P_x(T_0 < \infty) > 0$ on a set of positive measure. Let $G(x, A) = \int_0^\infty P_x(X_t \in A)\, dt$. Under the above conditions $G(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure and there is a version of the density $g(x)$ having the following properties: (i) g is bounded. (ii) g is continuous except perhaps at 0 where it may have a jump. (iii) $g(0) = g(0+) \wedge g(0-)$. Let $C = [g(0+) \vee g(0-)]^{-1}$ and let $h(x) = P(T_x < \infty)$. Then $h(x) = Cg(x)$. The above facts can be found in [1].

For $a > 0$ let $N(a)$ be the total occupation time of $[0, a]$ and let $R(a)$ be the Lebesgue measure of the range of the process intersected with $[0, a]$. Observe that

$$ER(a) = \int_0^a h(x)dx = C \int_0^a g(x)dx = CEN(a). \tag{1.1}$$

Our goal in this paper is to investigate the asymptotic behavior of $R(a)$ and $N(a)$ as $a \to \infty$.

Pruitt and Taylor [**6**] seem to have been the first to investigate the asymptotic behavior of $R(a)$ as $a \to \infty$. They examined the asymmetric Cauchy processes. If the asymmetry parameter β is -1 (so the process has no positive jumps) then $[0, \infty)$ is a transient set and $R(a) \uparrow R(\infty)$ a.s. In all other cases $[0, \infty)$ is a recurrent set and $R(a) \uparrow \infty$ a.s. In this case Pruitt and Taylor [**6**] showed that

$$ER(a) \sim k_1 \frac{a}{\ln a}, \qquad a \to \infty,$$

for a specified constant k_1, and found the remarkable fact that

$$R(a) \ln a / a \xrightarrow{D} R$$

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where R or $R-1$ is geometrically distributed, according as $\beta < 0$ or $\beta > 0$. In [**4**] the asymptotic behavior of $N(a)$ was determined for the asymmetric Cauchy processes. If $\beta = -1$, $N(a) \uparrow N(\infty) < \infty$ a.s. For $\beta > -1$, $N(a) \ln a/a \xrightarrow{D} (1/C)\, R$. Hence for $\beta > -1$ we have the curious fact that $N(a)/EN(a)$ and $R(a)/ER(a)$ have the same limiting distribution.

Let $\widetilde{X}_t$ be a stable process with exponent α and asymmetry parameter β. For $b \neq 0$, the process $X_t = \widetilde{X}_t + bt$ is the stable process with drift b. Suppose $\alpha < 1$. Then as for the asymmetric Cauchy processes, $[0, \infty)$ is a transient set if $\beta = -1$. In all other cases $[0, \infty)$ is a recurrent set. It was shown in [**4**] that for $\beta > -1$, $EN(a) \sim k_2 a^\alpha$, $a \to \infty$, for a specified numerical constant k_2, and that $N(a)/EN(a) \xrightarrow{D} N$, $a \to \infty$, where N is the total occupation time of $[0,1]$ by the process $\widetilde{X}_t$.

One of the motivations for the present paper was to find the limiting distribution for $R(a)/ER(a)$ for the stable processes with drift, exponent $\alpha < 1$, and $\beta > -1$. Another motivation was to understand why the limiting distributions of $R(a)/ER(a)$ and $N(a)/EN(a)$ were the same for the asymmetric Cauchy processes with $\beta > -1$. Remarkably, the following holds.

THEOREM 1. *Suppose* $[0, \infty)$ *is a recurrent set. Then*

$$E\left[\frac{R(a)}{ER(a)} - \frac{N(a)}{EN(a)}\right]^2 \to 0, \quad a \to \infty. \tag{1.2}$$

If $[0, \infty)$ is a transient set $R(a) \uparrow R(\infty)$ and $N(a) \uparrow N(\infty)$, $N(\infty)$ and $R(\infty)$ are $< \infty$ a.s., and generally there is no connection between the distributions of $R(\infty)$ and $N(\infty)$.

Theorem 1 shows that when $[0, \infty)$ is a recurrent set then $N(a)/EN(a)$ and $R(a)/ER(a)$ are asymptotically equal. This not only explains the equality of the limiting distributions for the asymmetric Cauchy processes with $\beta > -1$, but also yields a powerful tool for finding the limiting distributions for $R(a)/ER(a)$.

In general, it is usually quite difficult to find the limiting distribution of $R(a)/ER(a)$ by direct methods, even when considerable information is available about the asymptotic behavior of $g(x)$. In contrast, it is usually a routine application of the method of moments to determine the limiting distribution of $N(a)/EN(a)$ (provided it exists). See e.g. [**4**]. We will illustrate the use of Theorem 1 by finding the limiting distributions of $R(a)/ER(a)$ for the stable processes with drift and exponent $\alpha < 1$, and by rederiving the Pruitt-Taylor result.

Let X_t be a stable process with drift and exponent $\alpha < 1$. Assume $\beta > -1$. Let

$$k_2 = \left[1 + \beta^2 \tan\left(\frac{\pi\alpha}{2}\right)\right]^{-1} \Gamma\left(\frac{1-\alpha}{\pi}\right) \sin\left(\frac{\pi\alpha}{2}\right) \alpha^{-1}(1+\beta).$$

Then it was shown in [**4**] that $EN(a) \sim k_2 a^\alpha$, $a \to \infty$, and, as mentioned above, $N(a)/k_2 a^\alpha \xrightarrow{D} N$. Using these facts, (1.1), and Theorem 1, we can conclude the following.

THEOREM 2. *For a stable process with drift and exponent* $\alpha < 1$ *and* $\beta > -1$

$$R(a)/Ck_2 a^\alpha \xrightarrow{D} N.$$

Note: An explicit formula for C can be found in [**4**].

As a final illustration of the use of Theorem 1 let X_t be an asymmetric Cauchy process. Let Z be a geometrically distributed random variable with parameter $\rho = (1-\beta)/(1+\beta)$ if $\beta > 0$ and with parameter $(1+\beta)/(1-\beta)$ if $\beta < 0$. Again, by an easy application of the method of moments, it was shown in [**4**] that

$$\frac{N(a)}{a/\ln a} \xrightarrow{D} \frac{1}{C}(Z+1) \quad \text{if } \beta > 0$$

and

$$\frac{N(a)}{a/\ln a} \xrightarrow{D} \frac{Z}{C} \quad \text{if } \beta < 0.$$

Using Theorem 1 we can deduce the result due to Pruitt and Taylor [**6**] that

$$\frac{R(a)}{a/\ln a} \xrightarrow{D} (Z+1) \quad \text{if } \beta > 0 \quad \text{and} \quad \frac{R(a)}{a/\ln a} \xrightarrow{D} Z \quad \text{if } \beta < 0.$$

The proof given here via Theorem 1 seems to be substantially simpler than those given by Pruitt and Taylor [**5**, **6**].

To prove Theorem 1 we will need the renewal theorem for $g(x)$. Recall that a process is called type I transient except if it is transient with finite mean b. It is then called type II.

THEOREM 3. *Assume the process is type* I. *Then* $\lim_{|x|\to\infty} g(x) = 0$. *Assume the process is type* II *transient and* $b > 0$. *Then* $\lim_{x\to-\infty} g(x) = 0$ *and* $\lim_{x\to\infty} g(x) = 1/b$.

Both the stable processes with drift and exponent $\alpha < 1$ and the asymmetric Cauchy processes have infinite mean. It is only for Levy processes with infinite mean that there is any real problem with the asymptotic behavior of $N(a)$ and $R(a)$. If the process X_t has finite mean things are quite simple. Suppose $EX_t = bt$. If $b < 0$ then $[0, \infty)$ is a transient set since $X_t \to -\infty$ a.s. On the other hand if $b > 0$, $X_t \to \infty$ a.s. and $[0, \infty)$ is recurrent. Theorem 3 shows that for $b > 0$, $g(x) \to 1/b$ as $x \to \infty$ and $g(x) \to 0$ as $x \to -\infty$. Using this fact an easy computation shows that $E_0(N(a)/a - 1/b)^2 \to 0$. Thus by Theorem 1 $N(a)/EN(a)$ and $R(a)/ER(a)$ converge in L_2 to 1. This can be strengthened to a.s. convergence. That $N(a)/EN(a) \to 1$ a.s. is a routine application of the strong law of large numbers as we shall see. To establish that $R(a)/ER(a) \to 1$ a.s. we will need to extend a result due to Kesten, Spitzer, and Whitman, for Brownian motion on R^d to the Levy processes considered here. Since this fact actually holds in maximum generality we will take this opportunity to place the fact in the literature.

Let $\mathcal{G}$ be a second countable locally compact Abelian group and let X_t be an I.D. process on $\mathcal{G}$ such that the smallest closed subgroup of $\mathcal{G}$ containing all of the possible points of X_t (see p. 158 of [**3**] for the definition of a possible point) is $\mathcal{G}$, and assume X_t is transient. Let $C_B(t)$ be the Haar measure of the set $\{x: x \in X_s + B \text{ for some } s \le t\}$ where B is a compact set.

For Brownian motion on R^d, $d \ge 3$, Kesten, Spitzer, and Whitman showed that $\lim_{t\to\infty} C_B(t)/t = C(B)$ with probability one, where $C(B)$ is the capacity of B. This result was never published, but an outline of the proof can be found in [**2**, p. 252]. The full proof is in W. W. Whitman's 1964 Cornell University thesis.

Examination of the proof reveals that in fact the process X_t is Brownian motion if used in only two ways. (1) X_t is an I. D. process. (2) Let $T_B = \inf\{t > 0\colon X_t \in B\}$. Then

$$\frac{1}{t}\int P_x(T_B \le t)\,dx \to C(B), \quad t \to \infty, \tag{1.3}$$

and

$$\int P_x(T_B \le t, X_s \notin B \text{ for all } s > t)\;dx = tC(B). \tag{1.4}$$

In [**3**] a capacity theory was developed for all transient I. D. processes on $\mathcal{G}$ and the analogs of (1.3) and (1.4) were shown to be valid. ((1.3) is (11.3) of [**3**] and (1.4) is (11.14) of [**3**]). Consequently, the same proof given for the Brownian motion case shows the following holds.

THEOREM 4. *Let $C_B(t)$ be as above. Then* $\lim_{t\to\infty} C_B(t)/t = C(B)$ *with probability one.*

We will use Theorem 4 for $B = \{0\}$ to show the following.

THEOREM 5. *Let X_t be a type* II *transient process on R with mean $b > 0$. Then $N(a)/a \to 1/b$ with probability one and $R(a)/a \to C/b$ with probability one as $a \to \infty$. Also $N(a)/a$ and $R(a)/Ca$ converge in L_2 to $1/b$.*

Theorems 1 and 5 apply with obvious modifications to the occupation time $N(a_1, a_2]$ of $(a_1, a_2]$ and the Lebesgue measure of the range intersected with $(a_1, a_2]$ as $a_1 \to -\infty$, $a_2 \to \infty$. We omit the details.

More generally, let $\mathcal{G}$ be a second countable locally compact Abelian group and X_t a Levy process on $\mathcal{G}$ as described above. Assume X_t is transient and that $\{0\}$ is not essentially polar. Let $A_1 \subseteq A_2 \subseteq \cdots$, $\bigcup_n A_n = A$. Assume A_i are relatively compact and that A is recurrent. Let $N(A_i)$ be the occupation time of A_i and $R(A_i)$ the Haar measure of the range of the process intersected with A_i. Then the same arguments show that

$$E\left(\frac{N(A_i)}{EN(A_i)} - \frac{R(A_i)}{ER(A_i)}\right)^2 \to 0.$$

We omit the details.

2. Proofs.

PROOF OF THEOREM 3. We will establish the assertions of Theorem 3 for a type II process with mean $b > 0$. The same kind of argument can be used to establish the assertions of the theorem for a type I process. As usual, let C be the capacity of a point. Equation (7.6) of [**3**] shows $h(x+y) \ge h(x)h(y)$ or equivalently,

$$g(x+y) \ge Cg(x)g(y). \tag{2.1}$$

Let $a > 0$ and $h > 0$. Then (2.1) shows

$$\int_a^{a+h} g(y)dy \ge Cg(a)\int_0^h g(x)dx. \tag{2.2}$$

Using the renewal theorem (Theorem 10.1 of [1]) we find from (2.2) that

$$\overline{\lim_{a\to\infty}}\, g(a) \le \frac{1}{b}\left[\frac{C}{h}\int_0^h g(x)dx\right]^{-1}. \tag{2.3}$$

In all cases when $b > 0$, g is left continuous at 0 and $g(0+) = C^{-1}$. (See [1].) Thus letting $h \downarrow 0$ in (2.3) we find

$$\overline{\lim_{a\to\infty}}\, g(a) \le \frac{1}{b}. \tag{2.4}$$

Taking $x + y = a$ in (2.1) we find

$$g(a) \ge Cg(x)g(a-x).$$

Thus

$$hg(a) \ge C\int_{a-h}^a g(x)g(a-x)\, dx. \tag{2.5}$$

Since $g(0+) = C^{-1}$ we can, for any given $\varepsilon > 0$, find $h > 0$ such that $Cg(y) \ge 1-\varepsilon$ if $0 < y < h$. Thus for such h

$$g(a) \ge \frac{1-\varepsilon}{h}\int_{a-h}^a g(x)\, dx.$$

Using the renewal theorem again we find

$$\underline{\lim_{a\to\infty}}\, g(a) \ge \frac{1-\varepsilon}{b}. \tag{2.6}$$

It follows from (2.4) and (2.6) that $\lim_{a\to\infty} g(a) = 1/b$. The renewal theorem also shows

$$\lim_{a\to-\infty}\int_a^{a+h} g(x)dx = \lim_{a\to-\infty}\int_{a-h}^a g(x)dx = 0.$$

Using this fact the same kind of argument shows $\lim_{a\to-\infty} g(x) = 0$.

PROOF OF THEOREM 1. Observe that

$$EN(a)^2 = 2\int_0^a\int_0^a g(x)g(y-x)\, dx\, dy. \tag{2.7}$$

Now

$$\begin{aligned}
ER(a)^2/C^2 &= \frac{1}{C^2}\int_0^a\int_0^a P(T_x<\infty, T_y<\infty)\, dx\, dy \\
&= \frac{1}{C^2}\left[\int_0^a\int_0^a P(T_x<T_y<\infty) + P(T_y<T_x<\infty)\right] dx\, dy \\
&= \frac{1}{C^2}\int_0^a\int_0^a [P(T_x<T_y)P_x(T_y<\infty) + P(T_y<T_x)P_y(T_x<\infty)]\, dx\, dy \\
&\le \int_0^a\int_0^a [g(x)g(y-x) + g(y)g(x-y)]\, dx\, dy.
\end{aligned}$$

Thus

$$ER(a)^2/C^2 \le EN(a)^2. \tag{2.8}$$

Now

$$R(a)N(a) = \int_0^\infty dt \int_0^a dx\ 1_{[T_x<\infty]} 1_{[0,a]}(X_t).$$

Thus

$$\text{(2.9)} \qquad E[R(a)N(a)] = \int_0^\infty dt \int_0^a dx\ P(T_x < \infty,\ 0 \le X_t \le a).$$

But

$$\text{(2.10)} \qquad P(T_x < \infty,\ 0 < X_t \le a) = \int_0^t P(T_x \in ds) P_x(0 \le X_{t-s} \le a) + P(t < T_x < \infty,\ 0 \le X_t \le a).$$

Integrating (2.10) over t we find

$$\text{(2.11)} \quad \int_0^\infty P(T_x < \infty, 0 \le X_t \le a)\ dt = P(T_x < \infty) \int_0^a g(y-x) dy + \int_0^\infty P(t < T_x < \infty,\ 0 \le X_t \le a)\ dt.$$

Now

$$\int_0^\infty P(t < T_x < \infty, X_t \in [0,a])\ dt = \int_0^\infty \int_0^a P(T_x > t, X_t \in dy) P_y(T_x < \infty)\ dt.$$

Let $G_{\{x\}}(0, A) = \int_0^\infty P(T_x > t, X_t \in A)\, dt$ and let $G(x, A) = \int_0^\infty P_x(X_t \in A)\, dt$. The first passage equation shows

$$G_{\{x\}}(0, A) = G(0, A) - P(T_x < \infty) G(x, A).$$

Since $G(x, dy) = g(y-x)\, dy$ it follows that

$$\int_0^a \int_0^\infty P(T_x > t, X_t \in dy) P_y(T_x < \infty)\, dt = \int_0^a G_{\{x\}}(0, dy) P_y(T_x < \infty)$$
$$= \int_0^a g(y) P_y(T_x < \infty)\, dy - P(T_x < \infty) \int_0^a g(y-x) P_y(T_x < \infty)\, dy$$
$$= C \int_0^a [g(y) g(x-y)\, dy - C^2 g(x) g(y-x) g(x-y)]\, dy.$$

Using (2.11) and (2.9) we find

$$\text{(2.12)} \qquad E[R(a)N(a)] = C \int_0^a \int_0^a g(x) g(y-x)\ dx\ dy + C \int_0^a \int_0^a g(y) g(x-y)\ dx\ dy - C^2 \int_0^a \int_0^a g(x) g(y-x) g(x-y)\ dx\ dy.$$

Using (2.7), (2.8), and (2.12) we find

$$\text{(2.13)} \qquad E\left[\frac{R(a)}{C} - N(a)\right]^2 \le 2C \int_0^a \int_0^a g(x) g(y-x) g(x-y)\ dx\ dy.$$

Now (1.2) will hold provided we can show

$$\lim_{a\to\infty}(EN(a))^{-2}\int_0^a\int_0^a g(x)g(y-x)g(x-y)\,dx\,dy = 0. \tag{2.14}$$

The assumption that $[0,\infty)$ is a recurrent set implies that the process is either type I or type II with mean $b>0$. In either case $\lim_{x\to-\infty} g(x)=0$. Thus for given $\varepsilon>0$ we can find $\delta<\infty$ such that $g(-z)\le\varepsilon$ for all $z\ge\delta$. Let $\psi(x)=\int_0^x g(z)g(-z)dz$ and $M=\sup_x g(x)<\infty$. Now

$$\begin{aligned}\int_0^a\int_0^a g(x)g(y-x)g(x-y)\,dxdy &= \int_0^a g(a-x)\psi(x)\,dx+\int_0^a g(x)\psi(x)\,dx\\ &= \int_0^\delta [g(a-x)+g(x)]\psi(x)\,dx+\int_\delta^a [g(a-x)+g(x)]\psi(x)\,dx\\ &\le M^3\delta^2+2\varepsilon\left(\int_0^a g(x)\,dx\right)^2+2M^2\delta\int_0^a g(x)\,dx.\end{aligned}$$

Thus

$$\overline{\lim_{a\to\infty}}(EN(a))^{-2}\int_0^a\int_0^a g(x)g(y-x)g(x-y)\,dx\,dy\le 2\varepsilon.$$

Hence (2.14) holds.

PROOF OF THEOREM 5. The strong law of large number shows $X_t/t\to b$ as $t\to\infty$ with probability one. Fix $\varepsilon>0$. Then $\exists\, t_0(\omega)$ such that for $t\ge t_0(\omega)$, $(b-\varepsilon)t\le X_t(\omega)\le(b+\varepsilon)t$ for all ω except perhaps for ω in a set of probability 0. Assume $b-\varepsilon>0$ and $a>(b+\varepsilon)t_0$.

$$N(a)\ge\int_{t_0}^{a/b+\varepsilon} 1_{[0,a]}(X_t)\,dt=\frac{a}{b+\varepsilon}-t_0$$

and

$$N(a)\le\int_0^{a/b-\varepsilon} 1_{[0,a]}(X_t)\,dt\le\frac{a}{b-\varepsilon}$$

except for ω in the exceptional set. Hence

$$\frac{1}{b+\varepsilon}\le\underline{\lim_{a\to\infty}}\frac{N(a)}{a}\le\overline{\lim_{a\to\infty}}\frac{N(a)}{a}\le\frac{1}{b-\varepsilon}.$$

Thus $N(a)/a\to 1/b$ with probability one. Theorem 3 shows $EN(a)=\int_0^a g(x)dx$ $\sim a/b$ as $a\to\infty$.

Let R_t be the Lebesgue measure of $\{x\colon x=X_s$ for some $s\le t\}$. Then for ω not in the exceptional set

$$R(a)\le R_t$$

for all $t\ge(b-\varepsilon)\wedge t_0$. Assume $a/b-\varepsilon>t_0$. Then $R(a)\le R_{a/b-\varepsilon}$. For $B=\{0\}$, $\{X_s+B, s\le t\}=\{X_s, s\le t\}$. Thus by Theorem 4

$$\lim_{t\to\infty} R_t/t=C$$

with probability one. Hence for ω not in some set of probability 0.

$$\overline{\lim_{a\to\infty}}\frac{R(a)}{a}\le\lim_{a\to\infty}\frac{R_{a/b-\varepsilon}}{a}=\frac{C}{b-\varepsilon}.$$

Similarly

$$\varliminf_{a\to\infty}\frac{R(a)}{a}\geq\lim_{a\to\infty}\frac{R_{a/b+\varepsilon}-R_{t_0}}{a}=\frac{C}{b+\varepsilon}$$

except for ω in the exceptional set. Thus $\lim_{a\to\infty}R(a)/a=C/b$ with probability one. Since $0\leq R(a)/a\leq 1$, $ER(a)^2/(Ca)^2\to 1/b^2$. Thus $R(a)/Ca\overset{L_2}{\to}1/b$. Theorem 1 now shows that $N(a)/a\overset{L_2}{\to}1/b$.

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