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AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF A RANDOM POLYNOMIAL

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ABSTRACT. Let ν_n be the expected number of real zeros of a polynomial of degree n whose coefficients are independent random variables, normally distributed with mean 0 and variance 1. We find an asymptotic expansion for ν_n of the form

$$\nu_n = \frac{2}{\pi}\log(n+1) + \sum_{p=0}^{\infty} A_p (n+1)^{-p},$$

in which $A_0 = 0.625735818$, $A_1 = 0$, $A_2 = -0.24261274$, $A_3 = 0$, $A_4 = -0.08794067$, $A_5 = 0$. The numerical values of ν_n calculated from this expansion, using only the first four, or six, coefficients, agree with previously tabulated seven decimal place values ($1 \le n \le 100$) with an error of at most 10^{-7} when $n \ge 30$, or $n \ge 8$.

1. Introduction. Suppose that the coefficients of the polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ are independent, normally distributed random variables with mean 0 and variance 1. Kac [**1**] has shown that the expected number of real zeros of $f(z)$ is

$$\nu_n = \frac{4}{\pi}\int_0^1 \frac{\{1-h_n^2(x)\}^{1/2}}{1-x^2}\,dx, \tag{1}$$

in which $h_n(x) = (n+1)x^n(1-x^2)/(1-x^{2n+2})$. Kac [**1**] also showed that $\nu_n \sim (2/\pi)\log n$ for large n. Jamrom [**2, 3**] improved this result and showed that

$$\lim_{n=\infty}\{\nu_n - (2/\pi)\log(n+1)\} = A_0, \tag{2}$$

$$A_0 = \frac{2}{\pi}\left\{\log 2 + \int_0^1 (1-t^2\operatorname{csch}^2 t)^{1/2} t^{-1}\,dt - \int_1^\infty \{1-(1-t^2\operatorname{csch}^2 t)^{1/2}\} t^{-1}\,dt\right\}. \tag{3}$$

Wang [**4**] also derived (2); his value of A_0 is

$$\frac{8}{\pi}\int_0^1 (1-y^2)^{-1}\left\{1+\frac{(1-y^2-2y\log y)^{1/2}}{(1-y^2+2y\log y)^{1/2}}\right\}^{-1} dy.$$

(The substitution $y = e^{-t}$ and some simple manipulations show that the two values of A_0 are the same.)

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Several authors [**1**, **4**–**8**] have found upper and/or lower bounds for ν_n, all of which have been superseded, or anticipated, by the result of Yu [**9**] that $\nu_n - (2/\pi)\log(n+1)$ is an increasing sequence, so that

$$0.558728799 < \nu_1 - (2/\pi)\log 2 \le \nu_n - (2/\pi)\log(n+1) < A_0. \tag{4}$$

In this paper we will prove the following result.

THEOREM. *There is an asymptotic expansion of the form*

$$\nu_n \sim (2/\pi)\log(n+1) + \sum_{p=0}^{\infty} A_p (n+1)^{-p}, \tag{5}$$

in which A_0 *is defined by* (3), $A_1 = A_3 = A_5 = 0$, *and*

$$A_2 = -\frac{1}{3\pi}\int_0^\infty \{(1 - t^2\operatorname{csch}^2 t)^{-1/2} - 1\}t\,dt,$$
$$A_4 = -\frac{1}{180\pi}\int_0^\infty \{7 - 12(1 - t^2\operatorname{csch}^2 t)^{-1/2} + 5(1 - t^2\operatorname{csch}^2 t)^{-3/2}\}t^3\,dt.$$

The discovery that $A_1 = A_3 = A_5 = 0$ was unexpected. (We have been unable to decide the irresistible conjecture that $A_p = 0$ for all odd p.) We note that $A_0 = 0.625735818$, $A_2 = -0.24261274$, $A_4 = -0.08794067$. Our numerical value for A_0 does not agree with the value 0.6312 reported by Wang [**4**].

2. Preliminaries. In this section we perform certain algebraic manipulations useful for the subsequent analysis. The variable x is a number on the interval $(0, 1)$, and n and $N - 1$ are positive integers. It is convenient to assume that $n \ge 8$.

We define δ, a, t, w and μ so that $\delta = 1 - (n+1)^{-1/2}$, $a = (n+1)^{1/2}$, $x = 1 - t/(n+1)$, $x^n = e^{w-t}$, $x^{2n+2} = e^{-\mu-2t}$. Then $\delta \ge 2/3$, $a \ge 3$, $0 \le t \le a$ when $\delta \le x \le 1$, $w = t + n\log x$, $\mu = -2t - (2n+2)\log x$. It is now easy to verify the results stated in the following lemma.

LEMMA 1. *If* B, C, D *and* E *are defined so that*

$$B = e^{2w} - 1,\ C = t/2(n+1),\ D = (1 - e^{-\mu})/(e^{2t} - 1),$$
$$E = \frac{(2D - B + 2C + D^2 - C^2 + 2BC - BC^2)t^2\operatorname{csch}^2 t}{(1+D)^2(1 - t^2\operatorname{csch}^2 t)},$$

then

$$h_n^2(x) = 4t^2 e^{-2t}(1+B)(1-C)^2(1 - e^{-2t})^{-2}(1+D)^{-2},$$
$$1 - h_n^2(x) = (1 - t^2\operatorname{csch}^2 t)(1+E).$$

According to Taylor's Theorem, there is a number θ_1, depending on x and N, such that $0 < \theta_1 < 1$ and

$$\log x = -\sum_{p=1}^{N} p^{-1}t^p(n+1)^{-p} - (N+1)^{-1}t^{N+1}(n+1-\theta_1 t)^{-N-1}.$$

Because $n+1 \ge n+1-\theta_1 t \ge n+1-a \ge 2(n+1)/3$ when $0 \le t \le a$ and $n \ge 8$, the number v_N defined as $\{(n+1)/(n+1-\theta_1 t)\}^{N+1}$ is such that $1 \le v_N \le (3/2)^{N+1}$. It is now easy to see that the following lemma is true.

LEMMA 2. *If $\psi_{pN}(t)$ and $\phi_{pN}(t)$ are defined so that*

$$\psi_{pN} = 2/(p+1), \qquad \phi_{pN} = (p+1-pt)/p(p+1) \quad \text{when } p = 1, 2, \ldots, N-1,$$
$$\psi_{NN} = 2v_N/(N+1), \qquad \phi_{NN} = N^{-1} - nv_N t/(N+1)(n+1),$$

then

$$\mu = t\sum_{p=1}^{N} \psi_{pN}(t)t^p(n+1)^{-p}, \qquad w = \sum_{p=1}^{N} \phi_{pN}(t)t^p(n+1)^{-p}.$$

We furnish upper bounds for w and e^{2w} in the following lemma.

LEMMA 3. *If $0 \le x \le 1$ and $n \ge 8$, then $w \le 1 - 8\log(9/8) < 0.05774$, and $e^{2w} \le (8/9)^{16}e^2 < 1.1225$.*

If we consider $w = (n+1)(1-x) + n\log x$ as a function of x on $(0,1)$, we see that $dw/dx = -n - 1 + nx^{-1}$, $d^2w/dx^2 = -nx^{-2} < 0$. It follows that w attains its absolute maximum $w_{\max}$ when $x = n/(n+1)$, and that

$$w_{\max} = 1 + n\log\{1-(n+1)^{-1}\} = \sum_{p=1}^{\infty} \frac{1}{p(p+1)(n+1)^p}.$$

Hence $w_{\max}$ is a decreasing function of n. This remark and some arithmetic suffice to prove the lemma.

We can now find a lower bound for E.

LEMMA 4. *If $\delta \le x \le 1$ and $n \ge 8$, then $E \ge -0.6508$.*

It follows from Lemma 2 when $\delta \le x \le 1$ and $n \ge 8$ that

$$0 \le \frac{t^2}{n+1} \le \mu = \frac{t^2}{n+1}\left\{1 + \frac{2v_2 t}{3(n+1)}\right\} \le \frac{t^2}{n+1}\left\{1 + \frac{9t}{4(n+1)}\right\},$$
$$2D \ge (2\mu - \mu^2)/(e^{2t} - 1).$$

Moreover, $B = e^{2w} - 1 = 2w + 2w^2\exp(2\theta_2 w)$ for some θ_2 such that $0 < \theta_2 < 1$. It then follows from Lemma 3 that

$$2w \le B \le 2w + 2w^2\exp\{\max(2w, 0)\} \le 2w + 2.245w^2.$$

We infer from the definitions of w and t that

$$(n+1)^{-1} = 1 - n(n+1)^{-1} \ge \frac{w}{t} = 1 - \sum_{p=1}^{\infty} np^{-1}t^{p-1}(n+1)^{-p}$$
$$\ge 1 - \sum_{p=1}^{\infty} np^{-1}a^{-p-1} = -\frac{1}{2a} + \sum_{p=2}^{\infty} \frac{2}{(p^2-1)a^p}.$$

Hence $w \le t/(n+1)$, $-1/2a \le w/t \le 1/2a$, $w^2 \le t^2/4(n+1)$.

If $R = 2D - B + 2C + D^2 - C^2 + BC(2 - C)$, we now see that

$$\begin{aligned}-R \le &-\frac{2\mu-\mu^2}{e^{2t}-1} + (2w + 2.245w^2) - \frac{t}{n+1} - 0\\ &+ \frac{t^2}{4(n+1)^2} - \frac{2wt}{n+1}\left\{1 - \frac{t}{4(n+1)}\right\} \le -\frac{2t^2}{(n+1)(e^{2t}-1)}\\ &+ \frac{t^4}{(n+1)^2(e^{2t}-1)}\left\{1 + \frac{9t}{2(n+1)}\right\} + \frac{81t^2}{16(n+1)^2}\\ &+ \frac{2t}{n+1} + \frac{0.56125t^2}{n+1} - \frac{t}{n+1} + \frac{t^2}{4(n+1)^2}\\ &+ \frac{t^2}{(n+1)^{3/2}} = t^2(n+1)^{-1}\{P_0(t) + (n+1)^{-1/2} + P_1(t)(n+1)^{-1}\\ &\qquad + P_2(t)(n+1)^{-2} + P_3(t)(n+1)^{-3}\},\end{aligned}$$

in which

$$P_0(t) = t^{-1} - 2(e^{2t}-1)^{-1} + 0.56125, \quad P_1(t) = t^2(e^{2t}-1)^{-1} + \tfrac{1}{4},$$
$$P_2(t) = 4.5t^3(e^{2t}-1)^{-1}, \quad \text{and} \quad P_3(t) = 5.0625t^4(e^{2t}-1)^{-1}.$$

It is a straightforward task to estimate the maxima of the functions $P_l(t)$. We find that $P_0(t) \le 1.56125$, $P_1(t) \le 0.4120$, $P_2(t) \le 0.7996$, $P_3(t) \le 1.5124$. Because $n \ge 8$ and $D \ge 0$, it follows that $-R \le 1.9524t^2(n+1)^{-1}$, $-E \le 1.9524\chi(t)(n+1)^{-1}$, in which

$$\chi(t) = t^4(1 - t^2\operatorname{csch}^2 t)^{-1}\operatorname{csch}^2 t = t^4/(\sinh^2 t - t^2).$$

The validity of Lemma 4 is now a consequence of the inequality, $\sinh^2 t - t^2 \ge t^4/3$.

There is a number θ_3 such that $0 < \theta_3 < 1$ and

$$B = e^{2w} - 1 = \sum_{k=1}^{N} \frac{2^k b_{kN} w^k}{k!},$$

in which $b_{kN} = 1$ when $k = 1, 2, \ldots, N-1$, and $b_{NN} = \exp(2\theta_3 w) < 1.1225$. It now follows from Lemma 2 that

$$B = \sum_{p=1}^{N^2} \beta_{pN}(t)t^p(n+1)^{-p},$$

in which β_{pN} is a polynomial of degree p in t, whose coefficients are independent of both n and N, when $1 \le p \le N-1$, and β_{pN} is bounded when $0 \le t \le a$ by a polynomial of degree N in t, whose coefficients are independent of n, when $N \le p \le N^2$. Therefore,

$$B = \sum_{p=1}^{N} B_{pN}(t)t^p(n+1)^{-p},$$

in which $B_{pN} = \beta_{pN}$ when $1 \le p \le N-1$, and

$$B_{NN} = \sum_{p=N}^{N^2} \beta_{pN} t^{p-N}(n+1)^{-p+N},$$

so that B_{NN} is bounded when $0 \le t \le a$ by a polynomial of degree N in t whose coefficients are independent of n. In particular,

$$B_{1N} = 2 - t, \quad B_{2N} = \tfrac{1}{6}(18 - 16t + 3t^2), \quad B_{3N} = \tfrac{1}{6}(24 - 29t + 10t^2 - t^3),$$
$$B_{4N} = \tfrac{1}{360}(1800 - 2664t + 1280t^2 - 240t^3 + 15t^4),$$
$$B_{5N} = \tfrac{1}{360}(2160 - 3708t + 2224t^2 - 590t^3 + 70t^4 - 3t^5)$$

when $N \ge 6$.

In a similar manner we find that

$$D = \omega(t) \sum_{p=1}^{N} D_{pN}(t) t^p (n+1)^{-p},$$

in which $\omega(t) = t/(e^{2t} - 1) = t(\coth t - 1)/2$, and D_{pN} is a polynomial of degree $p - 1$ in t, whose coefficients are independent of n and N, when $1 \le p \le N - 1$, and D_{NN} is bounded when $0 \le t \le a$ by a polynomial of degree $N - 1$ in t whose coefficients are independent of n. In particular,

$$D_{1N} = 1, \quad D_{2N} = \tfrac{1}{6}(4 - 3t), \qquad D_{3N} = \tfrac{1}{6}(3 - 4t + t^2),$$
$$D_{4N} = \tfrac{1}{360}(144 - 260t + 120t^2 - 15t^3),$$
$$D_{5N} = \tfrac{1}{360}(120 - 264t + 170t^2 - 40t^3 + 3t^4,$$

when $N \ge 6$. It follows that

$$R = \sum_{p=1}^{N} R_{pN}(t) t^p (n+1)^{-p},$$

in which R_{pN} is independent of n and N and is bounded when $0 \le t < \infty$ by a polynomial of degree p in t when $1 \le p \le N - 1$. Also, R_{NN} is bounded when $0 \le t \le a$ by a polynomial of degree N in t whose coefficients are independent of n. In particular,

$$R_{1N} = 2\omega - 1 + t = t(\coth t - t^{-1}), \; R_{2N} = \omega^2 + \tfrac{1}{3}(4 - 3t)\omega - \tfrac{1}{12}(15 - 20t + 6t^2),$$
$$R_{3N} = \tfrac{1}{3}(4 - 3t)\omega^2 + \tfrac{1}{3}(3 - 4t + t^2)\omega - \tfrac{1}{12}(18 - 29t + 14t^2 - 2t^3),$$
$$R_{4N} = \tfrac{1}{36}(52 - 72t + 21t^2)\omega^2 + \tfrac{1}{180}(144 - 260t + 120t^2 - 15t^3)\omega$$
$$- \tfrac{1}{360}(630 - 1164t + 725t^2 - 180t^3 + 15t^4),$$
$$R_{5N} = \tfrac{1}{180}(264 - 510t + 280t^2 - 45t^3)\omega^3 + \tfrac{1}{180}(120 - 264t + 170t^2 - 40t^3 + 3t^4)\omega$$
$$- \tfrac{1}{360}(720 - 1479t + 1094t^2 - 365t^3 + 55t^4 - 3t^5),$$

when $N \ge 6$. The same sort of reasoning shows that

$$(1 + D)^{-2} = 1 + \omega(t) \sum_{p=1}^{N-1} u_{pN}(t) t^p (n+1)^{-p},$$

in which u_{pN} is independent of n and N and is bounded when $0 \le t < \infty$ by a polynomial of degree $p - 1$ in t when $1 \le p \le N - 2$, and $u_{N-1,N}$ is bounded when $0 \le t \le a$ by a polynomial of degree $N - 2$ in t whose coefficients are independent of n. (It is necessary to observe that $D \ge 0$.) In particular,

$$u_{1N} = -2, \quad u_{2N} = 3\omega - \tfrac{1}{3}(4 - 3t), \quad u_{3N} = -4\omega^2 + (4 - 3t)\omega - \tfrac{1}{3}(3 - 4t + t^2),$$
$$u_{4N} = 5\omega^3 - (8 - 6t)\omega^2 + \tfrac{1}{12}(52 - 72t + 21t^2)\omega - \tfrac{1}{180}(144 - 260t + 120t^2 - 15t^3),$$

when $N \ge 6$.

We next observe that

$$E = t^{-2}\chi(t)R(1+D)^{-2} = \chi(t)\sum_{p=1}^{N} E_{pN}(t)(n+1)^{-p}, \tag{6}$$

in which E_{pN} is independent of n and N and is bounded when $0 \le t < \infty$ by a polynomial of degree $2p-2$ in t when $1 \le p \le N-1$. (The bound is not obvious when $p=1$; it is true because $E_{1N} = t^{-1}R_{1N} = \coth t - t^{-1}$.) Moreover, E_{NN} is bounded when $0 \le t \le a$ by a polynomial of degree $2N-2$ in t whose coefficients are independent of n. In particular,

$$\begin{aligned}
E_{2N} &= -3\omega^2 + \tfrac{1}{3}(10-9t)\omega - \tfrac{1}{12}(15-20t+6t^2),\\
E_{3N} &= t\left\{4\omega^3 - (7-6t)\omega^2 + \tfrac{1}{6}(29-42t+14t^2)\omega - \tfrac{1}{6}(18-29t+14t^2-2t^3)\right\},\\
E_{4N} &= t^2\left\{-5\omega^4 + (12-10t)\omega^3 - \tfrac{1}{12}(145-216t+75t^2)\omega^2\right.\\
&\qquad + \tfrac{1}{60}(388-725t+420t^2-75t^3)\omega\\
&\qquad\qquad \left. -\tfrac{1}{360}(630-1164t+725t^2-180t^3+15t^4)\right\},\\
E_{5N} &= t^3\left\{6\omega^5 - \tfrac{1}{3}(55-45t)\omega^4 + \tfrac{1}{3}(73-110t+39t^2)\omega^3\right.\\
&\qquad - \tfrac{1}{60}(1094-2190t+1375t^2-270t^3)\omega^2\\
&\qquad + \tfrac{1}{180}(1479-3282t+2555t^2-825t^3+93t^4)\omega\\
&\qquad\qquad \left. -\tfrac{1}{360}(720-1479t+1094t^2-365t^3+55t^4-3t^5)\right\},
\end{aligned}$$

when $N \ge 6$.

We state our next result in this chain of manipulations as follows.

LEMMA 5. *There are functions $G_{pN}(t)$ such that*

$$(1+E)^{1/2} - 1 = \chi(t)\sum_{p=1}^{N} G_{pN}(t)(n+1)^{-p}.$$

The function G_{pN} is independent of n and N and is bounded when $0 \le t < \infty$ by a polynomial of degree $2p-2$ in t when $1 \le p \le N-1$. The function G_{NN} is bounded when $0 \le t \le a$ by a polynomial of degree $2N-2$ in t whose coefficients are independent of n. In particular,

$$\begin{aligned}
G_{1N} &= \tfrac{1}{2}E_{1N}, \quad G_{2N} = \tfrac{1}{2}E_{2N} - \tfrac{1}{8}\chi E_{1N}^2,\\
G_{3N} &= \tfrac{1}{2}E_{3N} - \tfrac{1}{4}\chi E_{1N}E_{2N} + \tfrac{1}{16}\chi^2 E_{1N}^3,\\
G_{4N} &= \tfrac{1}{2}E_{4N} - \tfrac{1}{8}\chi(E_{2N}^2 + 2E_{1N}E_{3N}) + \tfrac{3}{16}\chi^2 E_{1N}^2 E_{2N} - \tfrac{5}{128}\chi^3 E_{1N}^4,\\
G_{5N} &= \tfrac{1}{2}E_{5N} - \tfrac{1}{4}\chi(E_{1N}E_{4N} + E_{2N}E_{3N})\\
&\qquad + \tfrac{3}{16}\chi^2 E_{1N}(E_{2N}^2 + E_{1N}E_{3N}) - \tfrac{5}{32}\chi^3 E_{1N}^3 E_{2N} + \tfrac{7}{256}\chi^4 E_{1N}^5,
\end{aligned}$$

when $N \ge 6$.

This follows by a now familiar reasoning from (6) and the identity,

$$(1+E)^{1/2} - 1 = \sum_{k=1}^{N}\left\{\frac{\Gamma(k-\frac{1}{2})(-1)^{k-1}\gamma_{kN}}{2\pi^{1/2}k!}\right\}E^k,$$

in which $\gamma_{kN} = 1$ when $1 \le k \le N-1$ and $\gamma_{NN} = (1+\theta_4 E)^{-N+1/2}$, that is valid for some θ_4 such that $0 < \theta_4 < 1$. We need to observe that, as a consequence of Lemma 4, $1+\theta_4 E \ge 0.3492$, $\gamma_{NN} \le (2.8367)^{N-1/2}$.

3. Proof of the theorem. We observe first that

$$\nu_n = \frac{4}{\pi}\int_0^\delta (1-x^2)^{-1}\,dx - F + G,$$

$$\nu_n = (2/\pi)\log\{(1+\delta)/(1-\delta)\} - F + G, \tag{7}$$

in which

$$F = \frac{4}{\pi}\int_0^\delta (1-x^2)^{-1}\left[1-\{1-h_n^2(x)\}^{1/2}\right] dx$$

$$= \frac{4}{\pi}\int_0^\delta (1-x^2)^{-1} h_n^2(x)\left[1+\{1-h_n^2(x)\}^{1/2}\right]^{-1} dx,$$

$$G = \frac{4}{\pi}\int_\delta^1 (1-x^2)^{-1}\{1-h_n^2(x)\}^{1/2}\,dx.$$

We now show that F is asymptotically negligible.

LEMMA 6. *$F = O(a^3 e^{-2a}) = o\{(n+1)^{-N}\}$ for large n and any N.*

Here and throughout this section the constants implicit in the symbols O and o may depend on N, but are independent of n. Because $0 \le h_n(x) \le (n+1)x^n$,

$$0 < F < \frac{4(n+1)^2}{\pi(1-\delta^2)}\int_0^\delta x^{2n}\,dx = \frac{4(n+1)^2\delta^{2n+1}}{\pi(1-\delta^2)(2n+1)}.$$

We observe when $n \ge 8$ that

$$1-\delta^2 = a^{-1}(2-a^{-1}) \ge 5/3a, \quad 2(n+1)/(2n+1) \le 18/17,$$

and

$$\delta^{2n+1} = \exp\{(2n+1)\log(1-a^{-1})\} < \exp\{-(2n+1)a^{-1}\}$$

$$= \exp(-2a + a^{-1}) \le e^{1/3}e^{-2a}.$$

The assertion of the lemma is now apparent.

It we make the substitution $x = 1 - t/(n+1)$ in G and use Lemma 2, we see that

$$G = \frac{2}{\pi}\int_0^a \left[\{2(n+1)-t\}^{-1} + t^{-1}\right] dt + A_0 - H + I + J + K,$$

$$G = (2/\pi)\log\{a/(1+\delta)\} + A_0 - H + I + J + K, \tag{8}$$

in which A_0 is the constant defined in (3) and

$$H = \frac{2}{\pi}\int_0^a \{2(n+1)-t\}^{-1}\{1-(1-t^2\operatorname{csch}^2 t)^{1/2}\}\,dt,$$

$$I = \frac{2}{\pi}\int_0^a (1-t^2\operatorname{csch}^2 t)^{1/2} t^{-1}\{(1+E)^{1/2}-1\}\,dt,$$

$$J = \frac{2}{\pi}\int_0^a \{2(n+1)-t\}^{-1}(1-t^2\operatorname{csch}^2 t)^{1/2}\{(1+E)^{1/2}-1\}\,dt,$$

$$K = \frac{2}{\pi}\int_a^\infty \{1-(1-t^2\operatorname{csch}^2 t)^{1/2}\}t^{-1}\,dt.$$

We now show that K is asymptotically negligible.

LEMMA 7. *$K = O(ae^{-2a}) = o\{(n+1)^{-N}\}$ for large n and any N.*
It is obvious that $0 \le 1-(1-s)^{1/2} \le s$ when $0 \le s \le 1$. Therefore,

$$0 < K < \frac{2}{\pi}\int_a^\infty t \operatorname{csch}^2 t\, dt = O(ae^{-2a}).$$

An asymptotic expansion for H can be inferred from the following lemma.

LEMMA 8. *If H_p, L and M are defined so that*

$$H_p = (2^{1-p}\pi^{-1})\int_0^\infty \{1-(1-t^2\operatorname{csch}^2 t)^{1/2}\}t^{p-1}\, dt,$$

$$L = \sum_{p=1}^{N-1} 2^{1-p}\pi^{-1}(n+1)^{-p}\int_a^\infty \{1-(1-t^2\operatorname{csch}^2 t)^{1/2}\}t^{p-1}\, dt,$$

$$H = \sum_{p=1}^{N-1} H_p(n+1)^{-p} - L + M, \tag{9}$$

then $L = O(e^{-2a})$ and $M = O\{(n+1)^{-N}\}$ for large n.

The lemma follows from the definition of H, and Taylor's Theorem in the form

$$(1-C)^{-1} = \sum_{p=0}^{N-2} C^p + C^{N-1}(1+\theta_5 C)^{-N}, \qquad 0 < \theta_5 < 1, \tag{10}$$

if we observe that $1-\theta_5 C \ge 1 - C \ge 1 - 1/2a \ge 5/6$ when $n \ge 8$. In fact, $0 < M < (6/5)^N H_N(n+1)^{-N}$. Moreover,

$$0 < L < \sum_{p=1}^{N-1} 2^{1-p}\pi^{-1}(n+1)^{-p}\int_a^\infty t^{p+1}\operatorname{csch}^2 t\, dt$$
$$= \sum_{p=1}^{N-1}(n+1)^{-p}O(a^{p+1}e^{-2a}) = \sum_{p=1}^{N-1} O(a^{1-p}e^{-2a}) = O(e^{-2a}).$$

An asymptotic expansion for I can be inferred from the following lemma.

LEMMA 9. *If I_p, S and T are defined so that*

$$I_p = \frac{2}{\pi}\int_0^\infty t^3(1-t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t G_{pN}(t)\, dt,$$

$$S = \frac{2}{\pi}\sum_{p=1}^{N-1}(n+1)^{-p}\int_a^\infty t^3(1-t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t G_{pN}(t)\, dt,$$

$$T = \frac{2}{\pi}(n+1)^{-N}\int_0^a t^3(1-t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t G_{NN}(t)\, dt,$$

then I_p is independent of n and N when $1 \le p \le N-1$, $S = O(ae^{-2a})$ and $T = O\{(n+1)^{-N}\}$ for large n, and

$$I = \sum_{p=1}^{N-1} I_p(n+1)^{-p} - S + T. \tag{11}$$

The identity (11) is an immediate consequence of Lemma 5. Moreover, $G_{pN}(t) = O(t^{2p-2})$, uniformly in n when $n \geq 8$, for large t, if $1 \leq p \leq N-1$, so that

$$S = \sum_{p=1}^{N-1} (n+1)^{-p} \int_a^\infty O(t^{2p+1}e^{-2t})\,dt = \sum_{p=1}^{N-1} (n+1)^{-p} O(a^{2p+1}e^{-2a}) = O(ae^{-2a}).$$

Finally, the existence of the polynomial bound for G_{NN} described in Lemma 5 implies that $T = O\{(n+1)^{-N}\}$.

If we combine Lemma 5 and (10) we find that

$$\frac{(1+E)^{1/2}-1}{(n+1)(1-C)} = \chi(t) \sum_{p=2}^{N} Q_{pN}(t)(n+1)^{-p},$$

in which Q_{pN} is independent of n and N and is bounded when $0 \leq t < \infty$ by a polynomial of degree $2p-4$ in t, when $2 \leq p \leq N-1$, and Q_{NN} is bounded when $0 \leq t \leq a$ by a polynomial of degree $2N-4$ in t whose coefficients are independent of n. In particular,

$$Q_{2N} = \tfrac{1}{2}E_{1N}, \quad Q_{3N} = \tfrac{1}{2}E_{2N} + \tfrac{1}{4}tE_{1N} - \tfrac{1}{8}\chi E_{1N}^2,$$

$$Q_{4N} = \tfrac{1}{2}E_{3N} + \tfrac{1}{4}tE_{2N} + \tfrac{1}{8}t^2E_{1N} - \tfrac{1}{4}\chi(E_{2N} + tE_{1N}/4)E_{1N} + \tfrac{1}{16}\chi^2E_{1N}^3,$$

$$\begin{aligned} Q_{5N} = {} & \tfrac{1}{2}E_{4N} + \tfrac{1}{4}tE_{3N} + \tfrac{1}{8}t^2E_{2N} + \tfrac{1}{16}t^3E_{1N} \\ & - \tfrac{1}{8}\chi(E_{2N}^2 + 2E_{1N}E_{3N} + tE_{1N}E_{2N} + t^2E_{1N}^2/4) \\ & + \tfrac{1}{16}\chi^2E_{1N}^2(3E_{2N} + tE_{1N}/2) - \tfrac{5}{128}\chi^3E_{1N}^4, \end{aligned}$$

when $N \geq 6$. An argument almost identical to that used in the proof of Lemma 9 now suffices to establish the following result.

LEMMA 10. *If J_p, U and V are defined so that*

$$J_p = \pi^{-1} \int_0^\infty t^4(1 - t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t\, Q_{pN}(t)\,dt,$$

$$U = \pi^{-1} \sum_{p=2}^{N-1} (n+1)^{-p} \int_a^\infty t^4(1 - t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t\, Q_{pN}(t)\,dt,$$

$$V = \pi^{-1}(n+1)^{-N} \int_0^a t^4(1 - t^2\operatorname{csch}^2 t)^{-1/2}\operatorname{csch}^2 t\, Q_{NN}(t)\,dt,$$

then J_p is independent of n and N when $2 \leq p \leq N-1$, $U = O(e^{-2a})$ and $V = O\{(n+1)^{-N}\}$ for large n, and

$$J = \sum_{p=2}^{N-1} J_p(n+1)^{-p} - U + V. \tag{12}$$

If we combine (7), (8), (9), (11) and (12), we see that

$$\nu_n = \frac{2}{\pi}\log(n+1) + A_0 + \sum_{p=1}^{N-1} A_p(n+1)^{-p} + O\{(n+1)^{-N}\},$$

in which $A_p = -H_p + I_p + J_p$ (the hitherto undefined quantity $J_1 = 0$). This establishes the existence of the asymptotic expansion (5) of the Theorem, and prescribes the coefficients A_p. There remains only the straightforward, although tedious, verification of the specific results for A_1, A_2, A_3, A_4 and A_5 stated in the Theorem, based on the definitions of H_p, I_p and J_p given in Lemmas 8, 9 and 10. This verification is facilitated by the reduction formula

$$\int_0^\infty t^{2q+p+1}(1-t^2\operatorname{csch}^2 t)^{-q-1/2}(t\coth t - 1)Z(t)\operatorname{csch}^{2q+2} t\,dt$$
$$= \int_0^\infty t^{2q+p-1}(1-t^2\operatorname{csch}^2 t)^{-q+1/2}Z_1(t)\operatorname{csch}^{2q} t\,dt,$$

valid if $1 \leq q$, $Z(t)$ is continuously differentiable on $(0,\infty)$, there is a positive integer m such that $Z(t) = O(t^m)$ and $Z'(t) = O(t^m)$ for large t, $Z(t) = O(t^{2q-p})$ and $Z'(t) = O(t^{2q-p})$ for small t, and

$$Z_1(t) = \{(2q+p-2qt\coth t)Z(t) + tZ'(t)\}/(2q-1).$$

Kahaner [**10**] has tabulated seven decimal place values of ν_n when $n = 1(1)100$. The expansion (5), terminated after the term for which $p=2$ (or $p=3$), produces seven decimal place values that either agree with or exceed by at most 10^{-7} those of Kahaner when $n \geq 30$. Inclusion of the term for which $p=4$ (and $p=5$) produces the same agreement when $n \geq 8$. The error when $n=1$ is only 0.0008575. (There is a typographical error in the value ν_{98} reported by Kahaner. The correct value is 3.5510552.)

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