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CONGRUENCE AND ONE-DIMENSIONALITY OF METRIC SPACES

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ABSTRACT. Two subsets A and B of a metric space (X, d) are said to be congruent if there is a bijection between them which preserves the distance d. We show that if a separable locally compact metric space is such that no distinct subsets of cardinality 3 are congruent then its dimension is ≤ 1. We also show that the real line $\mathbb{R}$ can be given a compatible metric with this property.

1. Introduction. We say that two subsets A, B of a metric space (X, d) are congruent if there exists a bijection between them which preserves the distance d. Using the notion of congruence our previous result [**2**] says that a nonempty separable metrizable space is zero-dimensional if and only if it admits a metric relative to which no two distinct sets of cardinality 2 are congruent. Our objective in this note is to explore the case where the cardinality of the sets mentioned above is 3. We prove

THEOREM 1. *If X is a locally compact separable metric space having a metric d such that no two distinct subsets of X of cardinality* 3 *are congruent relative to d, then* $\dim(X) \leq 1$.

THEOREM 2. *The real line $\mathbb{R}$ can be given a compatible metric d so that* $(\mathbb{R}, d)$ *contains no two distinct congruent subsets of cardinality* 3.

2. Proof of the theorems. If $x_1, x_2 \in X$ are two distinct points of a metric space (X, d) we denote by $B(x_1, x_2)$ the bisector set defined as $\{y \colon y \in X$ and $d(x_1, y) = d(x_2, y)\}$ (cf. [**3**, **4**, or **5**]).

LEMMA 2.1. *Let X be a separable metrizable space. Then,* $\dim(X) \leq n$ *if and only if X has an admissible totally bounded metric d such that if B is a d-bisector set in X then* $\dim(B) \leq n - 1$.

PROOF. This is Theorem 1 of [**4**].

As a corollary to this lemma we prove

LEMMA 2.2. *Let (X, d) be a totally bounded metric space containing no two distinct congruent sets of cardinality* 3. *Then* $\dim(X) \leq 1$.

PROOF. If X is empty or if it contains only one point then $\dim(X) \leq 0$. So assume X contains at least two points and let $B = B(x_1, x_2)$ be a bisector set of

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X. We observe that the set B cannot contain more than one point since if y_1, $y_2 \in B$ were two distinct points then $\{x_1, y_1, y_2\}$ and $\{x_2, y_1, y_2\}$ would be two distinct congruent sets of cardinality 3 which is contrary to our hypothesis. Thus we have that $\dim(B) \leq 0$ for every bisector set of X and Lemma 2.1 implies that $\dim(X) \leq 1$ which was to be proved.

The proof of our Theorem 1 now follows as a corollary to Lemma 2.2. Since (X, d) is a locally compact separable metric space each point $x \in X$ is contained in a precompact open subset $U \subseteq X$ so that the restriction of d to U is totally bounded. Since the dimension is a local property we conclude that $\dim(X) \leq 1$.

To prove our Theorem 2 we need some notation. We denote by $\mathbb{R}^2$ the Euclidean plane with the usual Euclidean metric on it. Let I denote the group of all orientation preserving isometries of $\mathbb{R}^2$, i.e., the proper motions and let I^* be the group of all isometries of $\mathbb{R}^2$ including reflexions. Let P denote the graph $\{(x, x^2)\colon x \in \mathbb{R}\}$ of the parabola $y = x^2$ in $\mathbb{R}^2$ and let $P^+ \subseteq \mathbb{R}^2$ denote the following subset of it: $\{(x, x^2)\colon x > 0\}$. We observe that P^+ is homeomorphic to $\mathbb{R}$ and we shall prove that it has the desired property relative to the Euclidean metric of $\mathbb{R}^2$.

LEMMA 2.3. *Let $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ be two distinct subsets of P^+ of cardinality* 3. *Then these sets cannot be congruent.*

PROOF. We give an indirect proof, assuming they are congruent. Then, as we know (cf. [**1**]) the isometry between $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ extends uniquely to an element $T \in I^*$. Now we distinguish two cases: Case (1) where T is a proper motion. Let the points $A_i = (x_i, x_i^2)$ and $B_j(y_j, y_j^2)$ be indexed so that for their first coordinates x_i and y_j we have $0 < x_1 < x_2 < x_3$ and $0 < y_1 < y_2 < y_3$ respectively. Then, since T is order-preserving we have: $B_i = TA_i$ for $i = 1, 2, 3$. Without loss of generality we may assume $x_1 \leq y_1$ from which one easily deduces that $x_2 \leq y_2$ and $x_3 \leq y_3$. An elementary computation shows that the negatively taken tangent of the angle at the vertex A_2 of the triangle $\{A_1, A_2, A_3\}$ equals

$$a_{13}[1 + (x_1 + x_2)(x_2 + x_3)(1 + x_1^2 + 2x_1x_3 + x_3^2)]^{-1}$$

where $a_{13} = [(x_3 - x_1)^2 + (x_3^2 - x_1^2)^2]^{1/2}$ is the distance between A_1 and A_3. Since both of these quantities are preserved under T we obtain the equality

$$\begin{aligned} 1 + (x_1 + x_2)(x_2 + x_3)(1 + x_1^2 + 2x_1x_3 + x_3^2) \\ = 1 + (y_1 + y_2)(y_2 + y_3)(1 + y_1^2 + 2y_1y_3 + y_3^2). \end{aligned}$$

But since the expression on the left is an increasing function of x_1, x_2 and x_3 and since $y_1 \geq x_1$, $y_2 \geq x_2$, $y_3 \geq x_3$ the equality would imply $x_i = y_i$ for $i = 1, 2, 3$, which is impossible since the sets $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ are distinct. So it remains to investigate Case (2), where T is a reflexion. Let L be the line in $\mathbb{R}^2$ which is pointwise invariant under T. We observe that the set $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ has cardinality at least 4 and that it is contained in the intersection $P^+ \cap TP^+$. We also observe that the line L would intersect the parabola P in two points C_1, C_2 where $C_1 \in P^+$ and $C_2 \notin P^+$. Thus, the set $\{A_1, A_2, A_3, B_1, B_2, B_3, C_2\}$ with cardinality at least 5 would be in the intersection $P \cap TP$ which contradicts the fact that two quadratic curves have at most 4 common points. So we arrived at the desired contradiction in both cases which concludes the proof of our lemma.

Identifying the line $\mathbb{R}$ with the set P^+ and taking for d the restriction of the Euclidean metric of $\mathbb{R}^2$ to P^+ we obtain the proof of our Theorem 2.

REMARK. The fact that $\mathbb{R}$ can be remetrized in such a manner that no two distinct sets of cardinality 3 are congruent supports our belief that also the converse of Theorem 1 is true, namely that if a separable metrizable space has dimension ≤ 1 then it has a metric with this property. One way to show it would be to prove that the universal 1-dimensional subset of $\mathbb{R}^3$, the Sierpiński cube, has such a metric. In [**3**] is shown that there exists a subset of $\mathbb{R}$ containing no two distinct congruent subsets of cardinality 2 which is homeomorphic to the Cantor set C. We conjecture that there exists a set $S^* \subseteq \mathbb{R}^3$ homeomorphic to the Sierpiński set S such that for every nonidentical isometry T of $\mathbb{R}^3$ the cardinality of the intersection $S^* \cap TS^*$ is at most 2 if T is a proper motion and at most 3 if it is a reflexion. This would imply that the set S^* contains no two distinct congruent subsets of cardinality 3.

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