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# INFINITESIMAL CHARACTERIZATION OF HOMOGENEOUS BUNDLES

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ABSTRACT. Consider a principal bundle $Q(B, H)$ on a base $B$ which is compact and has finite fundamental group. We give necessary and sufficient conditions, in terms of the Atiyah sequence of $Q(B, H)$, for $Q(B, H)$ to be locally isomorphic to a bundle of the form $G(G/S, S)$ for $G$ a Lie group and $S$ a closed subgroup of $G$. The proof involves the careful integration of certain infinitesimal actions of a Lie algebra on $Q, B$ and the universal cover of $B$.

By a homogeneous bundle we mean a principal bundle of the form $G(G/H, H)$, where $G$ is a Lie group and $H$ is a closed subgroup. The infinitesimal properties of an arbitrary principal bundle $Q(B, H, p)$ are encoded in its *Atiyah sequence*

$$(*) \qquad \frac{Q \times \mathfrak{h}}{H} \rightarrowtail \frac{TQ}{H} \overset{p_*}{\twoheadrightarrow} TB$$

in which $TQ/H$ is the vector bundle on $B$ quotiented from $TQ$ by the action $Xh = T(R_h)(X)$, $p_*$ is the quotient of $T(p)\colon TQ \to TB$, and $(Q \times \mathfrak{h})/H$ is the Lie algebra bundle associated to $Q(B, H)$ by the adjoint representation. [**5**] gives a detailed account of the Atiyah sequence and its properties and, in particular, shows that all of the infinitesimal connection theory of $Q(B, H)$ takes place in the Atiyah sequence, rather than in the bundle itself. Further, the Atiyah sequence of a principal bundle behaves much like the Lie algebra of a Lie group: a morphism of Atiyah sequences integrates to a local morphism of principal bundles; a principal bundle $Q(B, H)$ with $Q$ connected has a universal covering (or *monodromy*) bundle $\widetilde{Q}(B, \widehat{H})$, where $\widetilde{Q}$ is the universal cover and $\widehat{H}$ is locally isomorphic to $H$, and the natural map $\widetilde{Q}(B, \widehat{H}) \twoheadrightarrow Q(B, H)$ induces an isomorphism of Atiyah sequences; there is a correspondence between reductions of a principal bundle and suitable subobjects of its Atiyah sequence. See [**5**] for an account of this theory and historical references.

The purpose of this paper is to give a partial answer to the following question, put to the author by E. Ruh in 1986: Given the Atiyah sequence $(*)$ of a principal bundle $Q(B, H)$, what properties of the Atiyah sequence ensure that the bundle is homogeneous? We answer this question provided that $B$ is compact and has finite fundamental group, showing that it is then necessary and sufficient that $TQ/H$ be trivializable as a vector bundle in such a way that the bracket of constant sections is constant, and that the constant vector fields on $Q$ commute with the fundamental vector fields. The result establishes that the monodromy bundle $\widetilde{Q}(B, \widehat{H})$ must be a homogeneous bundle; $Q(B, H)$ itself may be a quotient of this by a discrete

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normal subgroup of $\widehat{H}$. The proof relies on the integration of infinitesimal actions of Lie groups; it is at this point that the compactness assumptions are used.

It is important for our purposes that a principal bundle isomorphic to a homogeneous bundle should be homogeneous. We therefore have to broaden the definition slightly: a *homogeneous bundle* is a principal bundle $Q(B, H, p)$, together with a Lie group $G$ acting transitively to the left on $Q$ and $B$, such that $p$ is equivariant, the actions of $G$ and $H$ on $Q$ commute, and the action of $G$ on $Q$ is free, as well as transitive. In particular a homogeneous bundle is equivariant in the sense of [**4**].

The first section briefly describes the Atiyah sequence of a homogeneous bundle, the second gives the main result in the case of a simply connected base manifold, and the third gives the general result.

I am most grateful to E. Ruh for putting the original question, and to A. Weinstein for the crucial formula in 1.2. I also thank P. J. Higgins for some valuable comments.

**1. The Atiyah sequence of a homogenous bundle.** Consider a principal bundle $Q(B, H, p)$. Then $H$ acts on the tangent bundle $TQ \to Q$ by $Xh = T(R_h)(X)$, where $R_h: Q \to Q$ is the right translation corresponding to $h$, and the quotient is a vector bundle $TQ/H \to B$. We denote elements of $TQ/H$ by $\langle X \rangle$, where $X \in TQ$. Define $p_*: TQ/H \to TB$ by $p_*(\langle X \rangle) = T(p)(X)$; then $p_*$ is a surjective submersion of vector bundles over $B$. For $V \in \mathfrak{h}$, let $V^*$ denote the fundamental vector field $V^*(u) = T(h \mapsto uh)_1(V)$ on $Q$. Lastly, quotient the trivial bundle $Q \times \mathfrak{h} \to Q$ over the action of $H$, $(u, V)h = (uh, Adh^{-1}V)$; the result is a vector bundle $(Q \times \mathfrak{h})/H \to B$, whose elements we denote by $\langle u, V \rangle$, and the map $(Q \times \mathfrak{h})/H \to TQ/H$, $\langle u, V \rangle \mapsto \langle V^*(u) \rangle$ is an injective vector bundle morphism over $B$ making

$$\frac{Q \times \mathfrak{h}}{H} \rightarrowtail \frac{TQ}{H} \overset{p_*}{\twoheadrightarrow} TB \tag{1}$$

an exact sequence of vector bundles.

Sections of $TQ/H$ are in bijective correspondence with $H$-invariant vector fields on $Q$. Since the bracket of invariant vector fields is invariant, $\Gamma(TQ/H)$ acquires a bracket $[\ ,\ ]$ which is alternating, obeys the Jacobi identity, and has the properties

$$[X, fY] = f[X, Y] + p_*(X)(f)Y \tag{2}$$

$$p_*([X, Y]) = [p_*(X), p_*(Y)] \tag{3}$$

for $X, Y \in \Gamma(TQ/H)$, $f \in C(B)$. Here $C(B)$ is the ring of smooth functions on $B$.

From (3) it follows that the bracket on $\Gamma(TQ/H)$ restricts to $\Gamma((Q \times \mathfrak{h})/H)$. In terms of $H$-equivariant maps $Q \to \mathfrak{h}$ this bracket is

$$[V, W](u) = [V(u), W(u)]_R,$$

where $[\ ,\ ]_R$ is the bracket on $\mathfrak{h}$ induced from the right-invariant vector fields on $H$. In particular, $(Q \times \mathfrak{h})/H$ is a Lie algebra bundle.

A full account of this construction, with references, is given in [**5**, Appendix A]. We will need the following example shortly.

EXAMPLE 1.1. Consider a trivial bundle $B \times H(B, H)$. Then $T(B \times H)/H \cong TB \oplus (B \times \mathfrak{h})$, where $\oplus$ is the direct sum over $B$, and $(B \times H \times \mathfrak{h})/H$ is isomorphic to the trivial Lie algebra bundle $B \times \mathfrak{h}$. The bracket on $TB \oplus (B \times \mathfrak{h})$ is

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus \{X(W) - Y(V) + [V, W]_R\}$$

where $X(W)$ and $Y(V)$ denote Lie derivatives. Compare [**5**, III 3.21]. □

Now consider a bundle of the form $G(G/H, H, p)$, where $G$ is a Lie group and $H$ is a closed subgroup. Using the right trivialization of $TG$, one easily sees that $TG/H$ has a trivialization

$$\langle X \rangle \mapsto (gH, T(R_g - 1)(X)), \qquad \frac{TG}{H} \to (G/H) \times \mathfrak{g}$$

where $X \in T(G)_g$. The map $p_*$ now becomes

$$(G/H) \times \mathfrak{g} \to T(G/H), \qquad (x, X) \mapsto \overline{X}(x)$$

where $\overline{X}(x) = T(g \mapsto gx)_1(X)$ is the vector field on $G/H$ induced by $X$ via the action of $G$ on $G/H$.

The following formula is due to A. Weinstein.

PROPOSITION 1.2. *The bracket on* $\Gamma(TG/H)$ *is given in term of maps* $G/H \to \mathfrak{g}$ *by*

$$[X, Y] = \overline{X}(Y) - \overline{Y}(X) + [X, Y]^{\cdot}$$

*where* $X, Y: G/H \to \mathfrak{g}$. *Here* $[\ ,\ ]^{\cdot}$ *is the pointwise (right-hand) bracket and* $\overline{X}$ *is the vector field on* $G/H$ *defined by* $\overline{X}(x) = \overline{(X(x))}(x)$; *that is,* $\overline{X}(x) = T(g \mapsto gx)_1(X(x))$.

PROOF. Consider the morphism $\varphi$ from $G(G/H, H)$ to the trivial bundle $(G/H) \times G(G/H, G)$ defined by $\varphi(g) = (gH, g)$. Because this is a morphism of principal bundles, it induces a morphism of their Atiyah sequences $\varphi_*: (G/H) \times \mathfrak{g} \to T(G/H) \oplus (G/H \times \mathfrak{g})$. (See, for example [**5**, A§3].) Now because $\varphi$ induces the identity map $G/H \to G/H$, the first component of $\varphi_*(x, X)$ must be $\overline{X}(x)$, and it is easy to see that the second is $(x, X)$ itself. Briefly, we write $\varphi_*(X) = \overline{X} \oplus X$. Now apply the formula from 1.1. □

It is clear from 1.2 that the bracket of constant sections is constant. Indeed the constant sections of $TQ/H$ corresponding to this trivialization are those of the form $gH \mapsto \langle T(R_g)_1(X) \rangle$, where $X$ is a fixed element of $\mathfrak{g}$. These in turn correspond to the right-invariant vector fields on $G$. Since the fundamental vector fields on $G$ are precisely those left-invariant vector fields which correspond to elements of $\mathfrak{h} \leq \mathfrak{g}$, it is immediate that the "constant" fields (in the sense of 2.1) on $G$ commute with the fundamental vector fields.

**2. The case of a simply-connected base manifold.** Given a principal bundle $Q(B, H)$ with $Q$ disconnected one may choose any component $Q_0$ and obtain a reduction $Q_0(B, H')$, where $H' = \{h \in H | Q_0 h = Q_0\}$, and it is immediate that the inclusion $Q_0(B, H') \rightarrowtail Q(B, H)$ induces an isomorphism of Atiyah sequences. Further, for any $Q(B, H)$ with $Q$ connected there is a natural *monodromy bundle* $\widetilde{Q}(B, \widehat{H})$, where $\widetilde{Q}$ is the universal covering of $Q$, and $\widehat{H}$ is a certain extension $\pi_1 Q \rightarrowtail \widehat{H} \twoheadrightarrow H$, such that the covering projection $\widetilde{Q} \twoheadrightarrow Q$ is a morphism of principal bundles $\widetilde{Q}(B, \widehat{H}) \twoheadrightarrow Q(B, H)$ and induces an isomorphism of Atiyah sequences.

(See, for example, [**5**, II §6, III §6].) It therefore follows that we may suppose that we are given the Atiyah sequence

$$\frac{Q \times \mathfrak{h}}{H} \rightarrowtail \frac{TQ}{H} \twoheadrightarrow TB$$

of a bundle $Q(B, H)$ in which $Q$ is simply-connected.

Notice too that if $G$ is a simply-connected Lie group, $H$ a closed subgroup of $G$ and $D$ a discrete normal subgroup of $H$, which is not normal in $G$, then $G(G/H, H)$ and $G/D(G/H, H/D)$ will have the same Atiyah sequence, but only the first will be a homogeneous bundle. The bundles $SU(2)(S^2, U(1))$ and $SU(2)/\mathbf{Z}_n(S^2, U(1)/\mathbf{Z}_n)$ for $n > 2$ are examples. On the other hand, the monodromy bundle of a homogeneous bundle is homogeneous [**5**, II 6.5].

We now come to the main result.

THEOREM 2.1. *Let $Q(B, H, p)$ be a principal bundle with $Q$ simply-connected and $B$ compact and simply-connected. Suppose that $TQ/H$ is trivializable as a vector bundle in such a way that* (i) *the bracket of constant sections of $TQ/H$ is constant, and* (ii) *the constant vector fields on $Q$ (that is, the $H$-invariant vector fields corresponding to the constant sections of $TQ/H$) commute with the fundamental vector fields. Then $Q(B, H)$ is a homogeneous bundle.*

PROOF. Let the trivialization be $\varphi: TQ/H \cong B \times V$. Because the bracket of constant sections is constant, $V$ acquires a Lie algebra structure and we henceforth denote it by $\mathfrak{g}$. (Notice that the bracket of any two maps $B \to \mathfrak{g}$ is now determined by condition (2) of §1.)

Now $\varphi$ lifts to another isomorphism of vector bundles, $\Phi: TQ \to Q \times \mathfrak{g}$, which is $H$-equivariant (where $H$ acts on $Q \times \mathfrak{g}$ by $(u, X)h = (uh, X)$). Given $X \in \mathfrak{g}$, the corresponding constant vector field, denoted $\vec{X}$, is $\vec{X}(u) = \Phi^{-1}(u, X)$; it is $H$-invariant. A general result of Kumpera [**3**, §33] asserts that an invariant vector field on a principal bundle is complete iff its projection to the base is complete; since $B$ is here assumed compact, it follows that $\vec{X}$ is complete.

From well-known results of Palais, in the form given in [**2**, 3.1.3], we obtain an action $G \times Q \to Q$ with the property that

$$T(g \mapsto gu)_1(X) = \vec{X}(u)$$

for all $u \in Q$, $X \in \mathfrak{g}$, where $G$ is the simply-connected Lie group corresponding to $\mathfrak{g}$. Also, because $\Phi$ is injective, it follows that each $\mathfrak{g} \to T(Q)_u$, $X \mapsto \vec{X}(u)$, is injective, and therefore the action of $G$ is locally free. Lastly, each $\mathfrak{g} \to T(Q)_u$ is an isomorphism, and so the orbits of $G$ are open in $Q$; since $Q$ is connected it follows that the action is transitive. So we have a transitive, locally free action, and therefore each evaluation map $G \to Q$, $g \mapsto gu$ is a covering; since $Q$ is simply-connected, each evaluation map must be a diffeomorphism. Thus $G \times Q \to Q$ is a simply transitive action.

Next define $\mathfrak{g} \to \Gamma TB$, $X \mapsto \overline{X}$, by $\overline{X}(x) = (p_* \circ \varphi^{-1})(x, X)$. Again, each $\overline{X}$ is complete, and by integration we obtain an action $G \times B \to B$, with

$$T(g \mapsto gx)_1(X) = \overline{X}(x)$$

for $x \in B$, $X \in \mathfrak{g}$. Since $p_*$ is a surjective submersion, each map $\mathfrak{g} \to T(B)_x$, $X \mapsto \overline{X}(x)$ is surjective and so, as before, the action of $G$ is transitive. Lastly, $\overline{X}$ is the projection under $p: Q \to B$ of the $H$-invariant vector field $\overrightarrow{X}$ and it follows that $p$ is $G$-equivariant.

Return now to $Q$. We have a left $G$-action and a right $H$-action and wish to show that they commute. By assumption, $[\overrightarrow{X}, V^*] = 0$ for all $X \in \mathfrak{g}$, $V \in \mathfrak{h}$. Now the action of $G$ on $Q$ provides $\overrightarrow{X}$ with a global flow, namely $\theta_t(u) = \exp tXu$. From general considerations, $V^*$ has the global flow $\psi_s(u) = u \exp sV$. Commutativity of $\overrightarrow{X}$ and $V^*$ implies that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all $s, t \in \mathbf{R}$ and therefore

$$\exp tX(u \exp sV) = (\exp tXu) \exp sV$$

for all $u \in Q$. Now $G$ and $H$ are both connected, $G$ by assumption and $H$ by putting $\pi_0 Q = \pi_1 B = 0$ in the homotopy sequence of $Q(B, H)$. So $\exp(\mathfrak{g})$ generates $G$ and $\exp(\mathfrak{h})$ generates $H$ and the above equation now implies that $g(uh) = (gu)h$ for all $g \in G$, $u \in Q$, $h \in H$.

This completes the proof. □

If one wants a bundle of the form $G(G/S, S)$, where $S$ is a subgroup of $G$ and $G(G/S, S) \cong Q(B, H)$, one can proceed as follows. Fix $u_0 \in Q$ and $x_0 = p(u_0) \in B$. Let $S$ be the stability subgroup of $G$ at $x_0$. Then each $s \in S$ sends $u_0$ to $su_0 \in p^{-1}(x_0)$ and so there exists a unique $h \in H$ with $su_0 = u_0 h^{-1}$. Write $h = f(s)$. Because the $G$ and $H$ actions commute, it follows that $f: S \to H$ is a morphism, and it is easily seen to be an isomorphism. We now have a trio of maps $F: G \to Q$, $g \mapsto gu_0$; $f: S \to H$; $F_0: G/S \to B$, $gS \mapsto gx_0$. It is easily checked that they give an equivariant isomorphism of principal bundles $F(F_0, f): G(G/S, S) \to Q(B, H)$, using again the commutativity of the $G$ and $H$ actions. The smoothness of $f$ follows from that of $F$.

Concerning the topological assumptions on $Q$ and $B$, we already noted that the simple-connectivity of $Q$ is merely a procedural hypothesis. The condition that $B$ be simply-connected was introduced only to force $H$ to be connected, and we remove this in §3. However the requirement that $B$ be compact is essential to the method of proof, and we see no way of doing without it.

**3. Multiply-connected base manifolds.** The idea here is to lift the bundle to the universal cover of the base, and to show that the resulting homogeneous bundle induces a homogeneous bundle on the original base. We assume that we are given a principal bundle $Q(B, H, p)$, with $Q$ simply-connected, with $B$ and the universal cover $\widetilde{B}$ compact, and whose Atiyah sequence satisfies the conditions (i) and (ii) of 2.1. Denote the covering projection $\widetilde{B} \to B$ by $c$ and $\pi_1 B$ by $\pi$.

Lift $p: Q \to B$ to $\widetilde{p}: Q \to \widetilde{B}$. Explicitly, one can fix $u_0 \in Q$, and for any $u \in Q$, take a path in $Q$ from $u_0$ to $u$ and let $\widetilde{p}(u)$ be the homotopy class of the projection of this path under $p$. Writing $x_0 = p(u_0)$, one has $\widetilde{p}(u_0) = \widetilde{x}_0$, the class of the path in $B$ constant at $x_0$. Now $\widetilde{p}: Q \to \widetilde{B}$ is a principal bundle with group $H_0$, the identity component of $H$, and is a reduction of the inverse-image bundle $c^* Q(\widetilde{B}, H)$. Indeed one can express $c^* Q(\widetilde{B}, H)$ as $Q \times \pi(\widetilde{B}, H)$. Here the projection $Q \times \pi \to \widetilde{B}$ is $(u, \lambda) \mapsto \widetilde{p}(u)\lambda$ and the action of $H$ is $(u, \lambda)h = (uh, \partial(h)^{-1}\lambda)$, where $\partial$ is the natural morphism $H \to \pi_0 H \cong \pi$.

For the Atiyah sequence of $Q(\widetilde{B}, H_0)$, one sees firstly that as vector bundles there are isomorphisms

$$\begin{array}{ccccc} \dfrac{Q \times \mathfrak{h}}{H_0} & \rightarrowtail & \dfrac{TQ}{H_0} & \overset{p_*}{\twoheadrightarrow} & T\widetilde{B} \\ \wr\| & & \wr\| & & \wr\| \\ c^*\left(\dfrac{Q \times \mathfrak{h}}{H}\right) & \rightarrowtail & c^*\left(\dfrac{TQ}{H}\right) & \twoheadrightarrow & c^*(TB) \end{array}$$

Now the module of sections of an inverse-image bundle can be represented as a tensor product, $\Gamma(c^*(TQ/H)) \cong C(\widetilde{B}) \otimes_{C(B)} \Gamma(TQ/H)$. Here $C(\widetilde{B})$ and $C(B)$ are the rings of smooth functions and $\varphi \otimes X$ corresponds to $\varphi(X \circ c)$. In these terms, $p_*$ becomes $\varphi \otimes X \to \varphi \otimes p_*(X)$, regarding $\Gamma(c^*(TB))$ as $C(\widetilde{B}) \otimes_{C(B)} \Gamma TB$, and the bracket becomes

$$[\varphi \otimes X, \psi \otimes Y] = \varphi\psi \otimes [X,Y] + \varphi\overline{X}(\psi) \otimes Y - \psi\overline{Y}(\varphi) \otimes X$$

where $\overline{X}, \overline{Y}$ denote the $\pi$-invariant vector fields on $\widetilde{B}$ which correspond to $p_*(X)$, $p_*(Y) \in \Gamma TB \cong \Gamma^\pi T\widetilde{B}$. This formula is forced by equation (2) of §1.

It is clear that $TQ/H_0 \cong c^*(TQ/H) \cong \widetilde{B} \times \mathfrak{g}$. Further, the constant sections of $c^*(TQ/H)$ are precisely those of the form $1 \otimes X$, where $X$ is a constant section of $TQ/H$ and 1 is the function constant at 1. The bracket formula above now shows that it is still true that the bracket of constant sections is constant. Lastly, the constant vector fields on $Q$ arising from $TQ/H_0 \cong \widetilde{B} \times \mathfrak{g}$ are identical to the constant vector fields for $TQ/H \cong B \times \mathfrak{g}$, and the fundamental vector fields are also the same. Thus all the conditions of 2.1 apply to $Q(\widetilde{B}, H_0, \widetilde{p})$ and we obtain $Q(\widetilde{B}, H_0) \cong G(G/S, S)$ where $G$ acts transtively on $\widetilde{B}$ with stability subgroup $S$ at $x_0$.

We now want to show that the actions of $G$ and $\pi$ on $\widetilde{B}$ commute. To do this, consider the action of $G$ on $B$ induced by $TQ/H \cong B \times \mathfrak{g}$. As before, this action is transitive and makes $c$ equivariant. Because $G$ is simply-connected, there is a canonical lift of the action of $G$ on $B$ to $\widetilde{B}$; by uniqueness this must be the same action as obtained from $TQ/H_0 \cong \widetilde{B} \times \mathfrak{g}$ [**1**, I.9]. Now the canonical lift action certainly commutes with the action of $\pi$.

We now have bundles $Q(\widetilde{B}, H_0, \widetilde{p})$ and $\widetilde{B}(B, \pi, c)$ and we want to show that the composite $c \circ \widetilde{p}\colon Q \to B$ is a homogeneous bundle. Since $Q(\widetilde{B}, H_0)$ is equivariantly isomorphic to a $G(G/S, S)$, we can work directly with the latter. We now have the homogeneous bundle $G(G/S, S, \widetilde{p})$ and a bundle $G/S(B, \pi, c)$, where $G$ and $\pi$ commute on $G/S$, $G$ acts transitively on $B$, and $c$ is equivariant. We have basepoints $1 \in G$, $S \in G/S$, and $x_0 \in B$ with $\tilde{p}(1) = S$ and $c(S) = x_0$. Denote the action of $\pi$ on $G/S$ by $(gS, \lambda) \mapsto (gS)\lambda$.

Define $K = (c \circ \widetilde{p})^{-1}(x_0) \subset G$. Then $K$ is a subgroup of $G$. For example, if $k_1,\ k_2 \in K$ then $k_1S = S\lambda_1$, $k_2S = S\lambda_2$ for some $\lambda_1, \lambda_2 \in \pi$, and $k_1k_2S = k_1(S\lambda_2) = (k_1S)\lambda_2 = S\lambda_1\lambda_2$, so $k_1k_2 \in K$. Lastly the cosets $gK$, $g \in G$, are identical to the fibres of $c \circ \widetilde{p}$: for $g \in G$, $k \in K$ we have

$$\begin{aligned}(c \circ \widetilde{p})(gk) &= c(g(kS)) = c(g(S\lambda)) \quad (\text{for some } \lambda \in \pi) \\ &= c((gS)\lambda) = (c \circ \widetilde{p})(g),\end{aligned}$$

and if $g_1, g_2 \in G$ and $g_1 S = (g_2 S)\lambda$ for some $\lambda \in \pi$, then $g_1 S = g_2(S\lambda) = g_2(kS)$ for some $k \in K$, and hence $g_1 = g_2 ks$. It is clear that $S \subset K$. The same calculations in fact prove that $K$ is the stability subgroup of $G$ acting on $B$ with reference point $x_0$.

We therefore have that $c \circ \widetilde{p}\colon G \to B$ is (isomorphic to) a homogeneous bundle $G(G/K, K)$. This last part of the proof could be paraphrased as saying that the composite of a homogeneous bundle with an equivariant bundle is a homogeneous bundle.

Summarizing we have the following result.

THEOREM 3.1. *Let $Q(B, H, p)$ be a principal bundle with $Q$ simply-connected and $B$ compact with finite fundamental group. Suppose that $TQ/H$ is trivializable as a vector bundle in such a way that the conditions* (i) *and* (ii) *of* 2.1 *are satisfied. Then $Q(B, H)$ is a homogeneous bundle.* $\square$

Notice that in 2.1 we actually proved slightly more than we have here. In 2.1 the bundle $Q(B, H)$ acquired an action from $G$ which made it an equivariant bundle, and the isomorphism $G(G/S, S) \to Q(B, H)$ was an isomorphism of equivariant bundles. Here the actions of $G$ on $Q$ and $B$ need not make $Q(B, H)$ an equivariant bundle; all we assert is that there is a bundle isomorphism $G(G/S, S) \to Q(B, H)$.

It would be interesting to know whether (i) and (ii) are sufficient when the base is not compact.

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