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INFINITESIMAL CHARACTERIZATION OF HOMOGENEOUS BUNDLES

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ABSTRACT. Consider a principal bundle $Q(B,H)$ on a base B which is compact and has finite fundamental group. We give necessary and sufficient conditions, in terms of the Atiyah sequence of $Q(B,H)$, for $Q(B,H)$ to be locally isomorphic to a bundle of the form $G(G/S,S)$ for G a Lie group and S a closed subgroup of G. The proof involves the careful integration of certain infinitesimal actions of a Lie algebra on Q,B and the universal cover of B.

By a homogeneous bundle we mean a principal bundle of the form $G(G/H,H)$, where G is a Lie group and H is a closed subgroup. The infinitesimal properties of an arbitrary principal bundle $Q(B,H,p)$ are encoded in its *Atiyah sequence*

$$(*)\qquad \frac{Q\times\mathfrak{h}}{H} \rightarrowtail \frac{TQ}{H} \overset{p_*}{\twoheadrightarrow} TB$$

in which TQ/H is the vector bundle on B quotiented from TQ by the action $Xh = T(R_h)(X)$, p_* is the quotient of $T(p)\colon TQ \to TB$, and $(Q\times\mathfrak{h})/H$ is the Lie algebra bundle associated to $Q(B,H)$ by the adjoint representation. [**5**] gives a detailed account of the Atiyah sequence and its properties and, in particular, shows that all of the infinitesimal connection theory of $Q(B,H)$ takes place in the Atiyah sequence, rather than in the bundle itself. Further, the Atiyah sequence of a principal bundle behaves much like the Lie algebra of a Lie group: a morphism of Atiyah sequences integrates to a local morphism of principal bundles; a principal bundle $Q(B,H)$ with Q connected has a universal covering (or *monodromy*) bundle $\widetilde{Q}(B,\widehat{H})$, where $\widetilde{Q}$ is the universal cover and $\widehat{H}$ is locally isomorphic to H, and the natural map $\widetilde{Q}(B,\widehat{H}) \twoheadrightarrow Q(B,H)$ induces an isomorphism of Atiyah sequences; there is a correspondence between reductions of a principal bundle and suitable subobjects of its Atiyah sequence. See [**5**] for an account of this theory and historical references.

The purpose of this paper is to give a partial answer to the following question, put to the author by E. Ruh in 1986: Given the Atiyah sequence $(*)$ of a principal bundle $Q(B,H)$, what properties of the Atiyah sequence ensure that the bundle is homogeneous? We answer this question provided that B is compact and has finite fundamental group, showing that it is then necessary and sufficient that TQ/H be trivializable as a vector bundle in such a way that the bracket of constant sections is constant, and that the constant vector fields on Q commute with the fundamental vector fields. The result establishes that the monodromy bundle $\widetilde{Q}(B,\widehat{H})$ must be a homogeneous bundle; $Q(B,H)$ itself may be a quotient of this by a discrete

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normal subgroup of $\widehat{H}$. The proof relies on the integration of infinitesimal actions of Lie groups; it is at this point that the compactness assumptions are used.

It is important for our purposes that a principal bundle isomorphic to a homogeneous bundle should be homogeneous. We therefore have to broaden the definition slightly: a *homogeneous bundle* is a principal bundle $Q(B, H, p)$, together with a Lie group G acting transitively to the left on Q and B, such that p is equivariant, the actions of G and H on Q commute, and the action of G on Q is free, as well as transitive. In particular a homogeneous bundle is equivariant in the sense of [**4**].

The first section briefly describes the Atiyah sequence of a homogeneous bundle, the second gives the main result in the case of a simply connected base manifold, and the third gives the general result.

I am most grateful to E. Ruh for putting the original question, and to A. Weinstein for the crucial formula in 1.2. I also thank P. J. Higgins for some valuable comments.

1. The Atiyah sequence of a homogenous bundle. Consider a principal bundle $Q(B, H, p)$. Then H acts on the tangent bundle $TQ \to Q$ by $Xh = T(R_h)(X)$, where $R_h: Q \to Q$ is the right translation corresponding to h, and the quotient is a vector bundle $TQ/H \to B$. We denote elements of TQ/H by $\langle X \rangle$, where $X \in TQ$. Define $p_*: TQ/H \to TB$ by $p_*(\langle X \rangle) = T(p)(X)$; then p_* is a surjective submersion of vector bundles over B. For $V \in \mathfrak{h}$, let V^* denote the fundamental vector field $V^*(u) = T(h \mapsto uh)_1(V)$ on Q. Lastly, quotient the trivial bundle $Q \times \mathfrak{h} \to Q$ over the action of H, $(u, V)h = (uh, Adh^{-1}V)$; the result is a vector bundle $(Q \times \mathfrak{h})/H \to B$, whose elements we denote by $\langle u, V \rangle$, and the map $(Q \times \mathfrak{h})/H \to TQ/H$, $\langle u, V \rangle \mapsto \langle V^*(u) \rangle$ is an injective vector bundle morphism over B making

$$\frac{Q \times \mathfrak{h}}{H} \rightarrowtail \frac{TQ}{H} \overset{p_*}{\twoheadrightarrow} TB \tag{1}$$

an exact sequence of vector bundles.

Sections of TQ/H are in bijective correspondence with H-invariant vector fields on Q. Since the bracket of invariant vector fields is invariant, $\Gamma(TQ/H)$ acquires a bracket $[\ ,\ ]$ which is alternating, obeys the Jacobi identity, and has the properties

$$[X, fY] = f[X, Y] + p_*(X)(f)Y \tag{2}$$

$$p_*([X, Y]) = [p_*(X), p_*(Y)] \tag{3}$$

for $X, Y \in \Gamma(TQ/H)$, $f \in C(B)$. Here $C(B)$ is the ring of smooth functions on B.

From (3) it follows that the bracket on $\Gamma(TQ/H)$ restricts to $\Gamma((Q \times \mathfrak{h})/H)$. In terms of H-equivariant maps $Q \to \mathfrak{h}$ this bracket is

$$[V, W](u) = [V(u), W(u)]_R,$$

where $[\ ,\ ]_R$ is the bracket on $\mathfrak{h}$ induced from the right-invariant vector fields on H. In particular, $(Q \times \mathfrak{h})/H$ is a Lie algebra bundle.

A full account of this construction, with references, is given in [**5**, Appendix A]. We will need the following example shortly.

EXAMPLE 1.1. Consider a trivial bundle $B \times H(B, H)$. Then $T(B \times H)/H \cong TB \oplus (B \times \mathfrak{h})$, where $\oplus$ is the direct sum over B, and $(B \times H \times \mathfrak{h})/H$ is isomorphic to the trivial Lie algebra bundle $B \times \mathfrak{h}$. The bracket on $TB \oplus (B \times \mathfrak{h})$ is

$$[X \oplus V, Y \oplus W] = [X,Y] \oplus \{X(W) - Y(V) + [V,W]_R\}$$

where $X(W)$ and $Y(V)$ denote Lie derivatives. Compare [**5**, III 3.21]. □

Now consider a bundle of the form $G(G/H, H, p)$, where G is a Lie group and H is a closed subgroup. Using the right trivialization of TG, one easily sees that TG/H has a trivialization

$$\langle X \rangle \mapsto (gH, T(R_g - 1)(X)), \qquad \frac{TG}{H} \to (G/H) \times \mathfrak{g}$$

where $X \in T(G)_g$. The map p_* now becomes

$$(G/H) \times \mathfrak{g} \to T(G/H), \qquad (x, X) \mapsto \overline{X}(x)$$

where $\overline{X}(x) = T(g \mapsto gx)_1(X)$ is the vector field on G/H induced by X via the action of G on G/H.

The following formula is due to A. Weinstein.

PROPOSITION 1.2. *The bracket on* $\Gamma(TG/H)$ *is given in term of maps* $G/H \to \mathfrak{g}$ *by*

$$[X,Y] = \overline{X}(Y) - \overline{Y}(X) + [X,Y]^{\cdot}$$

where $X, Y: G/H \to \mathfrak{g}$. *Here* $[\ ,\ ]^{\cdot}$ *is the pointwise (right-hand) bracket and* $\overline{X}$ *is the vector field on* G/H *defined by* $\overline{X}(x) = \overline{(X(x))}(x)$; *that is,* $\overline{X}(x) = T(g \mapsto gx)_1(X(x))$.

PROOF. Consider the morphism φ from $G(G/H, H)$ to the trivial bundle $(G/H) \times G(G/H, G)$ defined by $\varphi(g) = (gH, g)$. Because this is a morphism of principal bundles, it induces a morphism of their Atiyah sequences $\varphi_*: (G/H) \times \mathfrak{g} \to T(G/H) \oplus (G/H \times \mathfrak{g})$. (See, for example [**5**, A§3].) Now because φ induces the identity map $G/H \to G/H$, the first component of $\varphi_*(x, X)$ must be $\overline{X}(x)$, and it is easy to see that the second is (x, X) itself. Briefly, we write $\varphi_*(X) = \overline{X} \oplus X$. Now apply the formula from 1.1. □

It is clear from 1.2 that the bracket of constant sections is constant. Indeed the constant sections of TQ/H corresponding to this trivialization are those of the form $gH \mapsto \langle T(R_g)_1(X) \rangle$, where X is a fixed element of $\mathfrak{g}$. These in turn correspond to the right-invariant vector fields on G. Since the fundamental vector fields on G are precisely those left-invariant vector fields which correspond to elements of $\mathfrak{h} \leq \mathfrak{g}$, it is immediate that the "constant" fields (in the sense of 2.1) on G commute with the fundamental vector fields.

2. The case of a simply-connected base manifold. Given a principal bundle $Q(B, H)$ with Q disconnected one may choose any component Q_0 and obtain a reduction $Q_0(B, H')$, where $H' = \{h \in H | Q_0 h = Q_0\}$, and it is immediate that the inclusion $Q_0(B, H') \rightarrowtail Q(B, H)$ induces an isomorphism of Atiyah sequences. Further, for any $Q(B, H)$ with Q connected there is a natural *monodromy bundle* $\widetilde{Q}(B, \widehat{H})$, where $\widetilde{Q}$ is the universal covering of Q, and $\widehat{H}$ is a certain extension $\pi_1 Q \rightarrowtail \widehat{H} \twoheadrightarrow H$, such that the covering projection $\widetilde{Q} \twoheadrightarrow Q$ is a morphism of principal bundles $\widetilde{Q}(B, \widehat{H}) \twoheadrightarrow Q(B, H)$ and induces an isomorphism of Atiyah sequences.

(See, for example, [**5**, II §6, III §6].) It therefore follows that we may suppose that we are given the Atiyah sequence

$$\frac{Q\times\mathfrak{h}}{H}\rightarrowtail\frac{TQ}{H}\twoheadrightarrow TB$$

of a bundle $Q(B,H)$ in which Q is simply-connected.

Notice too that if G is a simply-connected Lie group, H a closed subgroup of G and D a discrete normal subgroup of H, which is not normal in G, then $G(G/H,H)$ and $G/D(G/H,H/D)$ will have the same Atiyah sequence, but only the first will be a homogeneous bundle. The bundles $SU(2)(S^2,U(1))$ and $SU(2)/\mathbf{Z}_n(S^2,U(1)/\mathbf{Z}_n)$ for $n>2$ are examples. On the other hand, the monodromy bundle of a homogeneous bundle is homogeneous [**5**, II 6.5].

We now come to the main result.

THEOREM 2.1. *Let $Q(B,H,p)$ be a principal bundle with Q simply-connected and B compact and simply-connected. Suppose that TQ/H is trivializable as a vector bundle in such a way that* (i) *the bracket of constant sections of TQ/H is constant, and* (ii) *the constant vector fields on Q (that is, the H-invariant vector fields corresponding to the constant sections of TQ/H) commute with the fundamental vector fields. Then $Q(B,H)$ is a homogeneous bundle.*

PROOF. Let the trivialization be $\varphi\colon TQ/H\cong B\times V$. Because the bracket of constant sections is constant, V acquires a Lie algebra structure and we henceforth denote it by $\mathfrak{g}$. (Notice that the bracket of any two maps $B\to\mathfrak{g}$ is now determined by condition (2) of §1.)

Now φ lifts to another isomorphism of vector bundles, $\Phi\colon TQ\to Q\times\mathfrak{g}$, which is H-equivariant (where H acts on $Q\times\mathfrak{g}$ by $(u,X)h=(uh,X)$). Given $X\in\mathfrak{g}$, the corresponding constant vector field, denoted $\overrightarrow{X}$, is $\overrightarrow{X}(u)=\Phi^{-1}(u,X)$; it is H-invariant. A general result of Kumpera [**3**, §33] asserts that an invariant vector field on a principal bundle is complete iff its projection to the base is complete; since B is here assumed compact, it follows that $\overrightarrow{X}$ is complete.

From well-known results of Palais, in the form given in [**2**, 3.1.3], we obtain an action $G\times Q\to Q$ with the property that

$$T(g\mapsto gu)_1(X)=\overrightarrow{X}(u)$$

for all $u\in Q$, $X\in\mathfrak{g}$, where G is the simply-connected Lie group corresponding to $\mathfrak{g}$. Also, because Φ is injective, it follows that each $\mathfrak{g}\to T(Q)_u$, $X\mapsto\overrightarrow{X}(u)$, is injective, and therefore the action of G is locally free. Lastly, each $\mathfrak{g}\to T(Q)_u$ is an isomorphism, and so the orbits of G are open in Q; since Q is connected it follows that the action is transitive. So we have a transitive, locally free action, and therefore each evaluation map $G\to Q$, $g\mapsto gu$ is a covering; since Q is simply-connected, each evaluation map must be a diffeomorphism. Thus $G\times Q\to Q$ is a simply transitive action.

Next define $\mathfrak{g}\to\Gamma TB$, $X\mapsto\overline{X}$, by $\overline{X}(x)=(p_*\circ\varphi^{-1})(x,X)$. Again, each $\overline{X}$ is complete, and by integration we obtain an action $G\times B\to B$, with

$$T(g\mapsto gx)_1(X)=\overline{X}(x)$$

for $x \in B$, $X \in \mathfrak{g}$. Since p_* is a surjective submersion, each map $\mathfrak{g} \to T(B)_x$, $X \mapsto \overline{X}(x)$ is surjective and so, as before, the action of G is transitive. Lastly, $\overline{X}$ is the projection under $p: Q \to B$ of the H-invariant vector field $\overrightarrow{X}$ and it follows that p is G-equivariant.

Return now to Q. We have a left G-action and a right H-action and wish to show that they commute. By assumption, $[\overrightarrow{X}, V^*] = 0$ for all $X \in \mathfrak{g}$, $V \in \mathfrak{h}$. Now the action of G on Q provides $\overrightarrow{X}$ with a global flow, namely $\theta_t(u) = \exp tXu$. From general considerations, V^* has the global flow $\psi_s(u) = u \exp sV$. Commutativity of $\overrightarrow{X}$ and V^* implies that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all $s, t \in \mathbf{R}$ and therefore

$$\exp tX(u \exp sV) = (\exp tXu) \exp sV$$

for all $u \in Q$. Now G and H are both connected, G by assumption and H by putting $\pi_0 Q = \pi_1 B = 0$ in the homotopy sequence of $Q(B, H)$. So $\exp(\mathfrak{g})$ generates G and $\exp(\mathfrak{h})$ generates H and the above equation now implies that $g(uh) = (gu)h$ for all $g \in G$, $u \in Q$, $h \in H$.

This completes the proof. $\square$

If one wants a bundle of the form $G(G/S, S)$, where S is a subgroup of G and $G(G/S, S) \cong Q(B, H)$, one can proceed as follows. Fix $u_0 \in Q$ and $x_0 = p(u_0) \in B$. Let S be the stability subgroup of G at x_0. Then each $s \in S$ sends u_0 to $su_0 \in p^{-1}(x_0)$ and so there exists a unique $h \in H$ with $su_0 = u_0 h^{-1}$. Write $h = f(s)$. Because the G and H actions commute, it follows that $f: S \to H$ is a morphism, and it is easily seen to be an isomorphism. We now have a trio of maps $F: G \to Q$, $g \mapsto gu_0$; $f: S \to H$; $F_0: G/S \to B$, $gS \mapsto gx_0$. It is easily checked that they give an equivariant isomorphism of principal bundles $F(F_0, f): G(G/S, S) \to Q(B, H)$, using again the commutativity of the G and H actions. The smoothness of f follows from that of F.

Concerning the topological assumptions on Q and B, we already noted that the simple-connectivity of Q is merely a procedural hypothesis. The condition that B be simply-connected was introduced only to force H to be connected, and we remove this in §3. However the requirement that B be compact is essential to the method of proof, and we see no way of doing without it.

3. Multiply-connected base manifolds. The idea here is to lift the bundle to the universal cover of the base, and to show that the resulting homogeneous bundle induces a homogeneous bundle on the original base. We assume that we are given a principal bundle $Q(B, H, p)$, with Q simply-connected, with B and the universal cover $\widetilde{B}$ compact, and whose Atiyah sequence satisfies the conditions (i) and (ii) of 2.1. Denote the covering projection $\widetilde{B} \to B$ by c and $\pi_1 B$ by π.

Lift $p: Q \to B$ to $\widetilde{p}: Q \to \widetilde{B}$. Explicitly, one can fix $u_0 \in Q$, and for any $u \in Q$, take a path in Q from u_0 to u and let $\widetilde{p}(u)$ be the homotopy class of the projection of this path under p. Writing $x_0 = p(u_0)$, one has $\widetilde{p}(u_0) = \widetilde{x}_0$, the class of the path in B constant at x_0. Now $\widetilde{p}: Q \to \widetilde{B}$ is a principal bundle with group H_0, the identity component of H, and is a reduction of the inverse-image bundle $c^*Q(\widetilde{B}, H)$. Indeed one can express $c^*Q(\widetilde{B}, H)$ as $Q \times \pi(\widetilde{B}, H)$. Here the projection $Q \times \pi \to \widetilde{B}$ is $(u, \lambda) \mapsto \widetilde{p}(u)\lambda$ and the action of H is $(u, \lambda)h = (uh, \partial(h)^{-1}\lambda)$, where ∂ is the natural morphism $H \to \pi_0 H \cong \pi$.

For the Atiyah sequence of $Q(\widetilde{B}, H_0)$, one sees firstly that as vector bundles there are isomorphisms

$$\begin{array}{ccccc} \dfrac{Q\times\mathfrak{h}}{H_0} & \rightarrowtail & \dfrac{TQ}{H_0} & \overset{p_*}{\twoheadrightarrow} & T\widetilde{B} \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ c^*\left(\dfrac{Q\times\mathfrak{h}}{H}\right) & \rightarrowtail & c^*\left(\dfrac{TQ}{H}\right) & \twoheadrightarrow & c^*(TB) \end{array}$$

Now the module of sections of an inverse-image bundle can be represented as a tensor product, $\Gamma(c^*(TQ/H)) \cong C(\widetilde{B}) \otimes_{C(B)} \Gamma(TQ/H)$. Here $C(\widetilde{B})$ and $C(B)$ are the rings of smooth functions and $\varphi \otimes X$ corresponds to $\varphi(X \circ c)$. In these terms, p_* becomes $\varphi \otimes X \to \varphi \otimes p_*(X)$, regarding $\Gamma(c^*(TB))$ as $C(\widetilde{B}) \otimes_{C(B)} \Gamma TB$, and the bracket becomes

$$[\varphi \otimes X, \psi \otimes Y] = \varphi\psi \otimes [X, Y] + \varphi\overline{X}(\psi) \otimes Y - \psi\overline{Y}(\varphi) \otimes X$$

where $\overline{X}, \overline{Y}$ denote the π-invariant vector fields on $\widetilde{B}$ which correspond to $p_*(X)$, $p_*(Y) \in \Gamma TB \cong \Gamma^\pi T\widetilde{B}$. This formula is forced by equation (2) of §1.

It is clear that $TQ/H_0 \cong c^*(TQ/H) \cong \widetilde{B} \times \mathfrak{g}$. Further, the constant sections of $c^*(TQ/H)$ are precisely those of the form $1 \otimes X$, where X is a constant section of TQ/H and 1 is the function constant at 1. The bracket formula above now shows that it is still true that the bracket of constant sections is constant. Lastly, the constant vector fields on Q arising from $TQ/H_0 \cong \widetilde{B} \times \mathfrak{g}$ are identical to the constant vector fields for $TQ/H \cong B \times \mathfrak{g}$, and the fundamental vector fields are also the same. Thus all the conditions of 2.1 apply to $Q(\widetilde{B}, H_0, \widetilde{p})$ and we obtain $Q(\widetilde{B}, H_0) \cong G(G/S, S)$ where G acts transtively on $\widetilde{B}$ with stability subgroup S at x_0.

We now want to show that the actions of G and π on $\widetilde{B}$ commute. To do this, consider the action of G on B induced by $TQ/H \cong B \times \mathfrak{g}$. As before, this action is transitive and makes c equivariant. Because G is simply-connected, there is a canonical lift of the action of G on B to $\widetilde{B}$; by uniqueness this must be the same action as obtained from $TQ/H_0 \cong \widetilde{B} \times \mathfrak{g}$ [**1**, I.9]. Now the canonical lift action certainly commutes with the action of π.

We now have bundles $Q(\widetilde{B}, H_0, \tilde{p})$ and $\widetilde{B}(B, \pi, c)$ and we want to show that the composite $c \circ \widetilde{p}: Q \to B$ is a homogeneous bundle. Since $Q(\widetilde{B}, H_0)$ is equivariantly isomorphic to a $G(G/S, S)$, we can work directly with the latter. We now have the homogeneous bundle $G(G/S, S, \widetilde{p})$ and a bundle $G/S(B, \pi, c)$, where G and π commute on G/S, G acts transitively on B, and c is equivariant. We have basepoints $1 \in G$, $S \in G/S$, and $x_0 \in B$ with $\tilde{p}(1) = S$ and $c(S) = x_0$. Denote the action of π on G/S by $(gS, \lambda) \mapsto (gS)\lambda$.

Define $K = (c \circ \widetilde{p})^{-1}(x_0) \subset G$. Then K is a subgroup of G. For example, if k_1, $k_2 \in K$ then $k_1S = S\lambda_1$, $k_2S = S\lambda_2$ for some $\lambda_1, \lambda_2 \in \pi$, and $k_1k_2S = k_1(S\lambda_2) = (k_1S)\lambda_2 = S\lambda_1\lambda_2$, so $k_1k_2 \in K$. Lastly the cosets gK, $g \in G$, are identical to the fibres of $c \circ \widetilde{p}$: for $g \in G$, $k \in K$ we have

$$\begin{aligned}(c \circ \widetilde{p})(gk) &= c(g(kS)) = c(g(S\lambda)) \quad (\text{for some } \lambda \in \pi) \\ &= c((gS)\lambda) = (c \circ \widetilde{p})(g),\end{aligned}$$

and if $g_1, g_2 \in G$ and $g_1 S = (g_2 S)\lambda$ for some $\lambda \in \pi$, then $g_1 S = g_2(S\lambda) = g_2(kS)$ for some $k \in K$, and hence $g_1 = g_2 ks$. It is clear that $S \subset K$. The same calculations in fact prove that K is the stability subgroup of G acting on B with reference point x_0.

We therefore have that $c \circ \widetilde{p}\colon G \to B$ is (isomorphic to) a homogeneous bundle $G(G/K, K)$. This last part of the proof could be paraphrased as saying that the composite of a homogeneous bundle with an equivariant bundle is a homogeneous bundle.

Summarizing we have the following result.

THEOREM 3.1. *Let $Q(B, H, p)$ be a principal bundle with Q simply-connected and B compact with finite fundamental group. Suppose that TQ/H is trivializable as a vector bundle in such a way that the conditions* (i) *and* (ii) *of* 2.1 *are satisfied. Then $Q(B, H)$ is a homogeneous bundle.* □

Notice that in 2.1 we actually proved slightly more than we have here. In 2.1 the bundle $Q(B, H)$ acquired an action from G which made it an equivariant bundle, and the isomorphism $G(G/S, S) \to Q(B, H)$ was an isomorphism of equivariant bundles. Here the actions of G on Q and B need not make $Q(B, H)$ an equivariant bundle; all we assert is that there is a bundle isomorphism $G(G/S, S) \to Q(B, H)$.

It would be interesting to know whether (i) and (ii) are sufficient when the base is not compact.

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