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A CHARACTERIZATION OF LAŠNEV SPACES

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Dedicated to Professor Yukihiro Kodama on his sixtieth birthday

ABSTRACT. We give here a characterization of closed images of metrizable spaces in terms of the primitive concept of an ever finer sequence of partitions and a requirement considerably weaker than that of a k-network.

Almost twenty years after Lašnev's [**2**] first internal characterization of closed images of metrizable spaces came Foged's [**1**] improvement. Foged made use of the notion of k-networks of Michael [**3**] and came up with a much neater result. Nagata [**5, 6**], inspired by his student Sasaki's work, gave another solution in terms of g-functions. In all these three works, unmistakable is the motif of a σ-hereditarily closure-preserving (σ-HCP) collection of subsets. We give here a characterization of Lašnev spaces, as these closed images of metrizable spaces have come to be called, in terms of the much more primitive concept of an ever finer sequence of partitions and a requirement considerably weaker than that of a k-network, which, when in conjunction, one with the other, nonetheless become *at once* a σ-HCP k-network of subsets, the range of a g-function, into the set of all subsets instead of the topology, but otherwise of Nagata [**5, 6**] and an A-system of σ-HCP coverings the like of which we first encountered in Lašnev [**2**].

THEOREM. *A necessary and sufficient condition for a Fréchet T_2 space $(X, \mathscr{T})$ to be Lašnev is that there be a sequence $\langle \mathscr{P}_i \rangle$ of* ever finer *partitions of X (i.e., such a sequence of partitions that $\mathscr{P}_i$* refines *$\mathscr{P}_j$ if $i > j$) such that*

(†) given any $U \in \mathscr{T}$ and any sequence $\langle x_i \rangle$ (in U)[1] convergent to $\xi \in U$, there is a $P \in \mathscr{P} \equiv \bigcup\{\mathscr{P}_i : i \in \omega\}$ contained, along with its closure, in U and containing[1] some subsequence of $\langle x_i \rangle$.

PROOF. NECESSITY OF CONDITION. By the theorem of Foged [**1**], there is on the Lašnev space $(X, \mathscr{T})$ a σ-HCP *closed* k-network $\mathscr{F} = \bigcup\{\mathscr{F}_i : i \in \omega\}$ such that $\mathscr{F}_i \subset \mathscr{F}_{i+1}$. According to Nagata [**6**], a g-function $g: \omega \times X \to \mathscr{T}$ defined by the formula $g(n,x) = X \setminus \bigcup\{F \in \mathscr{F}_n : x \notin F\}$ satisfies $(1)_2$, $(2)_2$ and $(3)_2$ of Theorem 2 of [**6**]. Clearly, $g(i,x) \supset g(i+1,x)$, for all $i \in \omega$ and $x \in X$. For every $i \in \omega$ and every $x \in X$, writing $\mathscr{P}(i,x) \equiv \{g(i,x), \sim g(i,x)\}$, we define $\mathscr{P}_n$ to be

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[1] While it is clear what is meant by a sequence $\langle x_i \rangle$ being in U and P containing a subsequence of $\langle x_i \rangle$, we do not as a rule write $\langle x_i \rangle \subset U$ or say $\langle x_i \rangle \cap P$ or $\langle x_i \rangle \setminus P$. Here, we do. We treat $\langle x_i \rangle$ also as the set $\{x_i : i \in \omega\}$ as the relation $\subset$ or the operation $\cap$ or $\setminus$ demands. Results after the operations are of course sets which are then reverted to sequences in the obvious way.

$\bigwedge\{\mathscr{P}(i,x): i \leq n, x \in X\}$ and have thus defined an ever finer sequence $\langle\mathscr{P}_i\rangle$ of partitions of X. In Nagata's proof of the sufficiency part of Theorem 2 of **[6]**, one can see that our $\langle\mathscr{P}_i\rangle$ satisfies (†).

SUFFICIENCY OF CONDITION. Let there be a sequence $\langle\mathscr{P}_i\rangle$ of ever finer partitions of the Fréchet T_2 space $(X,\mathscr{T})$ with property (†). In the following, we write $\mathscr{H}(U,\langle x_i\rangle)$ for the family $\{P \in \mathscr{P}: \mathrm{Cl}\, P \subset U,\ P \text{ contains a subsequence of } \langle x_i\rangle\}$. In this notation, (†) merely says $\mathscr{H}(U,\langle x_i\rangle) \neq \varnothing$, for the given U and $\langle x_i\rangle$.

We first prove that $\overline{\mathscr{P}} \equiv \{\mathrm{Cl}\, P: P \in \mathscr{P}\}$ is a wcs-network **[6]** and therefore a k-network (when $\overline{\mathscr{P}}$ is σ-HCP): Given a $V \in \mathscr{T}$ and a sequence $\langle y_i\rangle$ in V convergent to $\eta \in V$, we are to prove that there is a finite subfamily $\mathscr{F}$ of $\mathscr{P}$ such that $\langle y_i\rangle$ is eventually in $\mathrm{Cl}\bigcup\mathscr{F} \subset V$. Note that by hypothesis there is a *maximal* $P_1 \in \mathscr{H}(V,\langle y_i\rangle)$. If $\langle y_i\rangle\backslash P_1$ [1] is a subsequence, there is a maximal $P_2 \in \mathscr{H}(V,\langle y_i\rangle\backslash P_1)$. Of course $P_2 \in \mathscr{H}(V,\langle y_i\rangle)$ and maximal there too. If $\langle y_i\rangle\backslash(P_1 \cup P_2)$ is a subsequence,.... This process however must stop at some stage N. For, otherwise, there are a sequence $\langle P_k\rangle$ of (disjoint) maximal members of $\mathscr{H}(V,\langle y_i\rangle)$, a subsequence $\langle z_k\rangle$ of $\langle y_i\rangle$ such that $z_k \in P_k$ for all $k \in \omega$, and a $Q \in \mathscr{H}(V,\langle z_k\rangle)$. For some $\kappa \in \omega$, $z_\kappa \in \langle z_k\rangle \cap Q$ [1] and we have $P_\kappa \subsetneqq Q$, a contradiction to the maximality of P_κ in $\mathscr{H}(V,\langle y_k\rangle)$. Clearly then the finite subfamily $\{P_n: 1 \leq n \leq N\}$ is a good choice for $\mathscr{F}$.

Note that the above proves that given $U \in \mathscr{T}$ and a sequence $\langle x_i\rangle$ (in U) convergent to $\xi \in U$, there are only *finitely* many maximal elements in $\mathscr{H}(U,\langle x_i\rangle)$. Let these be $P_1, P_2, \ldots, P_N$. Let $\mathscr{P}_\nu$ be the *first* partition to contain one of these. For $n = 1,2,\ldots,N$, let U_n be an open neighborhood of ξ such that $P_n \not\subset U_n$.[2] Let $V \equiv \bigcap\{U_n \cap U: 1 \leq n \leq N\}$. Clearly, there are again only *finitely* many maximal elements in $\mathscr{H}(V,\langle x_i\rangle \cap V)^1$, none of which appears in $\mathscr{P}_\nu$, while all of them are in $\mathscr{H}(U,\langle x_i\rangle)$. In other words,

$$\mathscr{H}(U,\langle x_i\rangle) \cap \bigcup\{\mathscr{P}_i: i \geq n\} \neq \varnothing$$

for arbitrarily large n.

Now we can see, for every $n \in \omega$, $\mathscr{P}_n$ is HCP. For, otherwise, there are a convergent sequence of points $\langle x_i\rangle$ and a sequence $\langle P_i\rangle$ of elements of $\mathscr{P}_n$ such that $x_i \in P_i$ for all $i \in \omega$, *when*, as noted immediately above, there is a $Q \in \bigcup\{\mathscr{P}_i: i \geq n\}$ containing a subsequence $\langle x_{i_k}\rangle$ of $\langle x_i\rangle$, from which it follows $Q \supsetneqq P_{i_1}$ which is impossible. Likewise, $\overline{\mathscr{P}}_n \equiv \{\mathrm{Cl}\, P: P \in \mathscr{P}_n\}$ is HCP. $\overline{\mathscr{P}}$ is therefore a σ-HCP k-network of closed subsets and X is Lašnev, according to Foged **[1]**.

APPLICATION. In view of the celebrated result of Stone **[7]** and Morita and Hanai **[4]**, the necessary and sufficient condition for Fréchet spaces to be Lašnev is also a characterization of metrizability among first countable spaces.

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[2]This is always possible unless $P_n = \{\xi\}$, in which case, the exercise of the present paragraph is hardly necessary.

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