\textbf{\textsc{S}\textsuperscript{\textsc{b}}-MANIFOLDS AND CONE-DUAL MAPS}

SARA DRAGOTTI AND GAETANO MAGRO

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\textbf{Abstract.} Let \( f: P \rightarrow Q \) be a simplicial map such that \( D(\alpha, f) \), the dual to \( \alpha \) with respect to \( f \), is a cone, for each simplex \( \alpha \) of \( Q \). It is shown that if \( P \) is an \( \mathcal{S} \)-manifold then \( f \) is approximable by PL-homeomorphisms, provided that \( f \) satisfies an extra condition on the boundary of \( P \).

\textbf{Introduction.} An interesting area of research has been that of trying to identify those maps which are approximable by PL-homeomorphisms or topological homeomorphisms. The domain and the range of these maps are spaces with extra structures, as PL-manifolds, homology manifolds, polyhedra etc. So, for example, a cellular map \( f: M \rightarrow N \) between topological \( n \)-manifolds is approximable by homeomorphisms (see Siebenmann [11], for \( n \neq 4, 5 \). The referee pointed out to us that Freedman and Edwards have proved the result for \( n = 4 \) and \( n = 5 \), respectively).

A generalization of this result to homology manifolds by introducing a more general concept of cellularity can be found in Henderson, [7].

If \( M \) is a PL-manifold, a cellular map, or PL-cellular map, it is not approximable by PL-homeomorphisms. A class of maps (transversely cellular maps) which do this is given by M. Cohen in [3].

In a recent work, [5], we have studied the problem when the domain is a homology manifold, and we have found a class of maps, called cone-dual maps, preserving homology manifold's structure, but nonpreserving the PL-homeomorphism's class. In the attempt to increase the last definition in order to obtain the required approximability, we arrive at defining the strong cone-dual maps. These last maps are approximable by PL-homeomorphisms when even the domain is a simple polyhedron.

In the present work we investigate the approximability by a PL-homeomorphism or top-homeomorphism, of maps between \( \mathcal{S} \)-manifolds. The \( \mathcal{S} \)-manifolds are a class of polyhedra which includes homology manifolds without boundary or with collared boundary.

The results obtained may be summarized as follows:

1. A cone-dual map \( f: M \rightarrow N \) is approximable by a PL-homeomorphism where \( M \) is an \( \mathcal{S} \)-manifold without boundary (Theorem 3.1).

2. If \( M \) is an \( \mathcal{S} \)-manifold with boundary \( \partial M \), a cone-dual map \( f: M \rightarrow N \) is approximable by a PL-homeomorphism provided \( f(\partial M) \) is collared in \( N \) (Theorem 3.2).

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(3) The condition "\(f(\partial M)\) is collared in \(N\)" is necessary to approximate \(f\) to a top-homeomorphism, i.e. there exist cone-dual maps \(f : M \rightarrow N\) where \(M\) is an \(\mathcal{Z}\)-manifold with boundary and \(N\) is not topological homeomorphic to \(M\) (§4).

1. Cone-dual maps. Let \(K\) be a simplicial complex, for each simplex \(\alpha\) of \(K\), the dual to \(\alpha\) in \(K\), denoted \(D(\alpha, K)\), and its subcomplex \(\hat{D}(\alpha, K)\) are defined by

\[
D(\alpha, K) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha < \sigma_1 < \cdots < \sigma_h < K \},
\]

and

\[
\hat{D}(\alpha, K) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha \leq \sigma_1 < \cdots < \sigma_h < K \}
\]

where \(b(\sigma_i)\) denotes the barycenter of \(\sigma_i\).

It is known that

(a) \(D(\alpha, K) = b(\alpha) \ast \hat{D}(\alpha, K)\)

(b) \(\hat{D}(\alpha, K) \cong_{PL} \text{Lk}(\alpha, K)\)

Throughout this paper all polyhedra are assumed to be compact and connected.

Let \(f : K \rightarrow L\) be a simplicial map, \(L'\) the first barycentric subdivision of \(L\), and \(K'\) a barycentric subdivision of \(K\) chosen so that \(f\) is also simplicial with respect to \(K'\) and \(L'\). For each simplex \(\alpha\) of \(L\), \(D(\alpha, f)\) and \(\hat{D}(\alpha, f)\) are the subcomplexes of \(K'\) defined by

\[
D(\alpha, f) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha < f(\sigma_1), \sigma_1 < \cdots < \sigma_h < K \},
\]

and

\[
\hat{D}(\alpha, f) = \{ b(\sigma_1) \cdots b(\sigma_h) | \alpha \leq f(\sigma_1), \sigma_1 < \cdots < \sigma_h < K \}.
\]

\(D(\alpha, f)\) is called the dual to \(\alpha\) with respect to \(f\).

We refer to [2, 3] for the proofs of following results.

**Proposition 1.1.** \(D(\alpha, f) = f^{-1}(D(\alpha, L)), \hat{D}(\alpha, f) = f^{-1}(\hat{D}(\alpha, L))\).

**Proposition 1.2.** \(K\) is the union of the duals of the simplexes of \(L\) with respect to \(f\). Moreover we have:

(a) \(\hat{D}(\alpha, f) = \bigcup_{\beta < \alpha} D(\beta, f)\),

(b) \(D(\alpha, f) \cap \hat{D}(\beta, f) = D(\alpha \cdot \beta, f)\),

where \(\alpha \cdot \beta\) is the simplex spanned by \(\alpha\) and \(\beta\) if there is one, \(\alpha \cdot \beta = \emptyset\) otherwise, and \(D(\emptyset, f) = \emptyset\).

**Proposition 1.3.** If \(\alpha^{i-1} < \alpha^i\), then \(D(\alpha^i, f)\) is a regular neighbourhood of \(f^{-1}(b(\alpha^i))\) in \(\hat{D}(\alpha^{i-1}, f)\) with boundary \(\hat{D}(\alpha^i, f)\). (For \(i = 0\) we assume \(\hat{D}(\alpha^{-1}, f) = K'\).)

We refer to [5] for the following definition and proposition.

A simplicial map \(f : K \rightarrow L\) is called strong cone-dual if \((D(\alpha, f), \hat{D}(\alpha, f))\) is a cone pair for each simplex \(\alpha\) of \(L\), i.e. there is a PL-homeomorphism of \(D(\alpha, f)\) onto a cone on \(\hat{D}(\alpha, f)\), which maps \(\hat{D}(\alpha, f)\) on \(\hat{D}(\alpha, f)\) (see Stallings [12]).

**Proposition 1.4.** If \(f : K \rightarrow L\) is a surjective strong cone-dual map, then there is a PL-homeomorphism \(\hat{f}\) of \(K\) into \(L\) such that \(\hat{f}(D(\alpha, f)) = D(\alpha, L)\), for each simplex \(\alpha\) of \(L\).

In [5] we have defined cone-dual maps between homology manifolds. It is possible to extend this definition to maps between polyhedra as soon as we define the boundary of a polyhedron.
Given an \( h \)-polyhedron \( P \), the boundary of \( P \) is the subpolyhedron defined inductively by

\[
\partial P = \begin{cases}
\emptyset, & h = 0, \\
\{ x \in P | \text{Lk}(x, P) = \text{point or } \partial \text{Lk}(x, P) \neq \emptyset \}, & h \geq 1,
\end{cases}
\]

**Definition 1.5.** A simplicial map \( f : P \to Q \) is called cone-dual with respect to the triangulations \( K \) and \( L \) of the polyhedra \( P \) and \( Q \) if \( D(\sigma, f) \) is a cone for each simplex \( \sigma \) of \( f(K) \), and \( D(\sigma, f/\partial K) \) is a cone for each simplex \( \sigma \) of \( f(\partial K) \).

The next theorem shows that Definition 1.5 does not depend on the triangulations chosen.

**Theorem 1.6.** Let \( f : K \to L \) be a cone-dual map, \( K \) and \( L \) triangulations of \( K \) and \( L \) such that \( f \) is also simplicial. Then \( f \) is cone-dual with respect to \( K \) and \( L \).

To prove this theorem we will use the following lemma.

**Lemma 1.7.** Let \( f : K \to L \) be a cone-dual map, if \( \beta \) is an \( i \)-simplex of \( f(K) \) and \( \beta' \) is a \( j \)-face of \( \beta \), then a regular nbd of \( f^{-1}(b(\beta')) \) in \( D(\beta', f) \) is PL-homeomorphic to a cone. (If \( \beta' = \emptyset \) assume \( D(\beta', f) = K' \).

**Proof.** If \( j = i - 1 \), the result follows from Proposition 1.3. So we suppose \( j < i - 1 \).

Let \( \beta^{i+1}, \beta^{i+2}, \ldots, \beta^{i-1} \) be a finite sequence of simplexes of \( f(K) \) such that \( \beta^i < \beta^{i+1} < \cdots < \beta^{i-1} < \beta' \). By Proposition 1.3, \( D(\beta'h, f) \) is a regular nbd of \( f^{-1}(b(\beta'h)) \) in \( D(\beta'^1, f) \) with boundary \( D(\beta'h, f) \). Hence \( D(\beta'h, f) \) is bicollared in \( D(\beta'^1, f) \). This implies that a regular nbd \( U \) of \( D(\beta'^1, f) \) in \( D(\beta'^1, f) \) is PL-homeomorphic to \( D(\beta'^1, f) \times [-1, 1]^r \) \( (r = i - j - 1) \), by a PL-homeomorphism

\[
\varphi : D(\beta'^1, f) \times [-1, 1]^r \to U
\]

so that \( \varphi(D(\beta'^1, f) \times \{0\}^r) = D(\beta'^1, f) \). Since \( D(\beta'^1, f) \) is a regular nbd of \( f^{-1}(b(\beta')) \) in \( D(\beta'^1, f) \), a regular nbd of \( f^{-1}(b(\beta')) \) in \( D(\beta', f) \) is PL-homeomorphic to \( D(\beta', f) \times [-1, 1]^r \), which is a cone. \( \square \)

**Proof of Theorem 1.6.** First suppose that \( \bar{L} \) is obtained from \( L \) by starring at only point \( v = b(\bar{v}) \). Generally for each simplex \( \alpha \) of \( \bar{L} \), by \( \bar{\alpha} \) we mean the carrier of \( \alpha \) in \( L \). Moreover we denote by \( \bar{D}(\alpha, f) \) and by \( \bar{\bar{D}}(\alpha, f) \) the corresponding carrier of \( D(\alpha, f) \) and \( \bar{D}(\alpha, f) \) with respect to \( f : \bar{K} \to \bar{L} \), i.e.: \( \bar{D}(\alpha, f) = f^{-1}(D(\alpha, \bar{L})) \), \( \bar{\bar{D}}(\alpha, f) = f^{-1}(\bar{D}(\alpha, \bar{L})) \).

To prove that \( \bar{D}(\alpha, f) \) is a cone for each \( \alpha \) of \( \bar{L} \), we proceed to consider the various cases.

**Case 1.** Assume \( \alpha \in \bar{L} \cap L \).

Note that if \( \alpha \) does not lie in \( \text{Lk}(v, \bar{L}) \), we have \( D(\alpha, \bar{L}) = D(\alpha, L) \), hence \( \bar{D}(\alpha, f) = D(\alpha, f) \).

If \( \alpha \) is a face of some simplex which contains \( v \), then there exists a PL-homeomorphism between \( \bar{D}(\alpha, f) \) and \( D(\alpha, f) \). In fact let \( \varphi(b(\tau)) = b(\bar{\tau}) \) for each vertex \( b(\tau) \)

\( ^1 \)An equivalent definition of boundary can be found in [13].
of $\overline{D}(\alpha, f)$. Evidently, if $\alpha$, in $\overline{L}$, is a face of $f(\tau)$, then $\alpha$, as simplex of $L$, will be a face of $f(\tilde{\tau})$. Hence $b(\tilde{\tau})$ is a vertex of $D(\alpha, f)$. It follows that $\varphi$ carries the vertices of $\overline{D}(\alpha, f)$ into vertices of $D(\alpha, f)$.

$\varphi$ is an injective map. In fact if $\varphi(b(\tau)) = \varphi(b(\delta))$, then $\tau$ and $\delta$ are simplexes of $K$ so that $\tilde{\tau} = \tilde{\delta}$. This implies either $\tau = \delta$, or $v$ lies in $\alpha$ ($\alpha < f(\tau), \alpha < f(\delta)$). Since $\alpha \in \overline{L} \cap L$, the last eventuality does not occur.

Trivially $\varphi$ is surjective, and hence bijective.

Observe that if $\tau_1 < \tau_2$, then $\tilde{\tau}_1 < \tilde{\tau}_2$. Hence if $b(\tau_1), \ldots, b(\tau_h)$ lie in a simplex of $\overline{D}(\alpha, f)$, then $b(\tilde{\tau}_1), \ldots, b(\tilde{\tau}_h)$ lie in a simplex of $D(\alpha, f)$. Therefore $\varphi$ can be extended to a PL-homeomorphism, which we will denote again by $\varphi$, of $\overline{D}(\alpha, f)$ onto $D(\alpha, f)$.

Observe that, in this case, if $\alpha \not\subset f(\tau)$ then $\alpha \not\subset f(\tilde{\tau})$. Consequently $\varphi$ takes $\overline{D}(\alpha, f)$ onto $D(\alpha, f)$, and $\varphi$ coincides with identity when $\alpha$ does not lie in the closure of the star of $v$ in $\overline{L}$.

Thus if $\alpha \in L \cap \overline{L}$, then $\overline{D}(\alpha, f)$, being PL-homeomorphic to the cone $D(\alpha, f)$, is a cone.

**Case II.** Assume $\alpha \in \overline{L} - L$, $\alpha \neq v$.

In this case $v$ is a vertex of $\alpha$. Since $\alpha \neq v$, there exists a simplex $\beta \in L \cap \overline{L}$ so that $\beta$ is a 1-codimensional face of $\alpha$ ($\beta$ = opposite face to $v$). Then $\overline{D}(\alpha, f)$ is a regular nbd of $f^{-1}(b(\alpha))$ in $\overline{D}(\beta, f)$. From the previous case it follows that $\overline{D}(\beta, f)$ is PL-homeomorphic to $D(\beta, f)$. On the other hand $f^{-1}(b(\alpha))$ is PL-homeomorphic to $f^{-1}(b(\tilde{\alpha}))$. Now observe that $\beta$ is also a face of $\tilde{\alpha}$, but in general it is not a 1-codimensional face. However, by Lemma 1.7, we can assert that $f^{-1}(b(\tilde{\alpha}))$ has in $\overline{D}(\beta, f)$ a regular nbd which is a cone. Thus, from the uniqueness theorem for regular nbd, it follows that $\overline{D}(\alpha, f)$ is a cone.

**Case III.** Assume $\alpha = v$.

By Lemma 1.7, $f^{-1}(v) = f^{-1}(b(\tilde{v}))$ has a regular nbd which is a cone in $K'$. As above, using the uniqueness theorem for regular nbd, and the fact that $K'$ is also a subdivision of $K$, we have that $\overline{D}(\alpha, f)$ is a cone.

In order to prove the result in the general case, we must show that, assuming $L$ and $\overline{L}$ as above, we can exchange their roles with respect to $f$. That is, if we suppose that $\overline{D}(\alpha, f)$ is a cone for every simplex $\alpha$ of $\overline{L}$, then $D(\alpha, f)$ is a cone for every $\alpha \in L$.

In fact, if $\alpha \in \overline{L} \cap L$, we have proved that $D(\alpha, f)$ is PL-homeomorphic to $\overline{D}(\alpha, f)$, and hence $D(\alpha, f)$ is a cone. If instead $\alpha$ lies in $L - \overline{L}$, then it is the carrier of a simplex $\alpha$ of $\overline{L}$ of the same dimension. Then, if $\beta$ is a simplex of $L \cap \overline{L}$ and a 1-codimensional face of $\alpha$ and $\tilde{\alpha}$, reasoning as before (Case II), we have that $f^{-1}(b(\alpha))$ has a regular nbd in $\overline{D}(\beta, f)$ which is a cone. Hence $D(\alpha, f)$ is a cone.

Finally, to complete the proof, it suffices to recall that equivalent triangulations of $L$ have a common subdivision, and that every subdivision $L'$ of $L$ can be obtained from $L$ by a finite number of subdivisions $L' = L_h \prec L_{h-1} \prec \cdots \prec L_1 = L$ so that $L_i$ is obtained from $L_{i-1}$ by introducing an only vertex. $\square$

2. **Duals and $\mathcal{L}$-manifolds.** In this section we investigate the dual structure induced by a simplicial map $f: K \to L$, on $K$, when $K$ is an $\mathcal{L}$-manifold.

For the reader's convenience, we reproduce here the definition of $\mathcal{L}$-manifold, according to [1].
Suppose we are given a class \( \mathcal{L}_n \), for each \( n \geq 0 \), of \((n - 1)\)-polyhedra (closed under PL-homeomorphisms), which satisfies:

1. Each member of \( \mathcal{L}_n \) is a polyhedron whose links lie in \( \mathcal{L}_{n-1} \).
2. \( \mathcal{L}_{n-1} \subseteq \mathcal{L}_n \) (i.e. the suspension of an \((n - 1)\)-link is an \( n\)-link).
3. \( c\mathcal{L}_{n-1} \cap \mathcal{L}_n = \emptyset \) (i.e. the cone on \((n - 1)\)-link is never a link).

Then an \( \mathcal{L}_n \)-manifold \( M \) is a polyhedron whose links lie either in \( \mathcal{L}_n \) or \( c\mathcal{L}_n \). The boundary of \( M \), \( \partial M \), consists of points whose links lie in the latter class.

As an immediate consequence of the definition we observe that the boundary of an \( \mathcal{L}_n \)-manifold \( M \) is itself an \( \mathcal{L}_{n-1} \)-manifold \( \partial M \) without boundary. Furthermore \( \partial M \) is collared in \( M \).

**REMARK 2.1.** The link of an \( i \)-simplex in an \( \mathcal{L}_n \)-manifold lies either in \( \mathcal{L}_{n-i} \) or in \( c\mathcal{L}_{n-i} \).

**REMARK 2.2.** Every polyhedron of \( \mathcal{L}_n \) is an \( \mathcal{L}_{n-1} \)-manifold without boundary.

**REMARK 2.3.** An \( \mathcal{L} \)-manifold, which is a cone, is a cone on the complete boundary.

In fact, let \( M = v \ast H \) be an \( \mathcal{L} \)-manifold. Since \( H = \text{Lk}(v, M) \), \( H \) lies either in \( \mathcal{L}_n \) or in \( c\mathcal{L}_{n-1} \). If \( H \) lies in \( \mathcal{L}_n \), then \( \partial H = \emptyset \). Hence \( H \) is the complete boundary of \( M \). If instead \( H = cX \), with \( X \in \mathcal{L}_{n-1} \), then we have: \( M = cX \cong \partial \Sigma X \) and \( \partial(c\Sigma X) = \Sigma X \cong \partial M \). This implies \( M \cong \partial M \).

**THEOREM 2.4.** Let \( f: M \rightarrow L \) be a simplicial map, where \( M \) is an \( \mathcal{L}_n \)-manifold and \( L \) a polyhedron. For each \( i \)-simplex \( \gamma \) of \( f(M) \), \( D(\gamma, f) \) is an \( \mathcal{L}_{n-1} \)-manifold with boundary \( \hat{D}(\gamma, f) \cup D(\gamma, f/\partial M) \).

**PROOF.** Let \( \sigma = b(\sigma_0) \cdots b(\sigma_h) \) be a simplex of \( D(\gamma, f) - \hat{D}(\gamma, f) \), we have that (see [3])

\[
\text{Lk}(\sigma, D(\gamma, f)) = \{b(\tau_0) \cdots b(\tau_q)| \gamma = f(\tau_0) = \cdots = f(\tau_q); \tau_q < \sigma_0 \} \\
= [((f/\sigma_0)^{-1}(b(\gamma))) \ast \hat{D}(\sigma_0, \hat{\sigma}_1) \ast \cdots \ast \hat{D}(\sigma_{h-1}, \hat{\sigma}_h) \ast \hat{D}(\sigma_h, M)]
\]

Let \( \dim \sigma_h = n_h \), using results about duals in a PL-manifold, it follows that \( \text{Lk}(\sigma, D(\gamma, f)) = \bigvee^r \text{Lk}(\sigma_h, M) \), where \( r = n_h - h - i \). Therefore \( \text{Lk}(\sigma, D(\gamma, f)) \) lies either in \( \mathcal{L}_{n-i-h} \) or \( c\mathcal{L}_{n-i-h-1} \), depending on whether \( \sigma_h \) (and hence \( \sigma \)) is in \( M - \partial M \) or not.

Suppose now that \( \sigma \) lies in \( \hat{D}(\gamma, f) \), we have

\[
\text{Lk}(\sigma, D(\gamma, f)) = D(\gamma, f/\sigma_0) \ast \hat{D}(\sigma_0, \hat{\sigma}_1) \ast \cdots \ast \hat{D}(\sigma_{h-1}, \hat{\sigma}_h) \ast \hat{D}(\sigma_h, M).
\]

Since \( D(\gamma, f/\sigma_0) \) is a PL-ball (see [3]), it follows that

\[
\text{Lk}(\sigma, D(\gamma, f)) \cong_{\text{PL}} v \ast \bigvee^{k+1} \text{Lk}(\sigma_h, M), \quad \text{where } k = n_h - i - h - 2.
\]

If \( \sigma_h \) does not lie in \( \partial M \), then \( \text{Lk}(\sigma_h, M) \) lies in \( \mathcal{L}_{n-n_h-1} \) and hence \( \text{Lk}(\sigma_h, D(\gamma, f)) \) lies in \( c\mathcal{L}_{n-i-h-1} \). If \( \sigma_h \) lies in \( \partial M \), then \( \text{Lk}(\sigma_h, M) \) is a cone on a polyhedron \( X \) of \( \mathcal{L}_{n-n_h-1} \), it follows that

\[
\text{Lk}(\sigma, D(\gamma, f)) \cong_{\text{PL}} v \ast c\bigvee^{k+1} X \cong_{\text{PL}} c\bigvee^{k+2} X.
\]

So we have that \( D(\gamma, f) \) is an \( \mathcal{L}_{n-i} \)-manifold.
Moreover $\text{Lk}(\sigma, D(\gamma, f))$ is a cone if and only if $\sigma$ lies either in $\partial M$ or in $\hat{D}(\gamma, f)$. This implies that

$$\partial D(\gamma, f) = \hat{D}(\gamma, f) \cup D(\gamma, f/\partial M).$$

3. Cone-dual maps and $\mathcal{L}$-manifolds. Let $f: M \to N$ be a cone-dual map. In [5] we have proved that if $M$ is a homology manifold, then $f(M)$ is itself a homology manifold. But in general $f(M)$ is not PL-homeomorphic to $M$.

The next theorem shows that this result can be improved if we suppose that $M$ is a homology manifold, or, more generally an $\mathcal{L}$-manifold, without boundary.

**THEOREM 3.1.** Let $f: M \to N$ be a surjective cone-dual map. If $M$ is an $\mathcal{L}$-manifold without boundary, then $F$ is approximable by a PL-homeomorphism.

**PROOF.** By Proposition 1.4 and Theorem 1.6 it suffices to prove that $f$ is a strong cone-dual map.

Let $\eta$ be a simplex of $N$. By hypothesis $D(\eta, f)$ is PL-homeomorphic to a cone $cX$. On the other hand, from Theorem 2.4, $D(\eta, f)$ is an $\mathcal{L}$-manifold with boundary $\hat{D}(\eta, f)$. Hence, using Remark 2.3, we can suppose $X = \hat{D}(\eta, f)$. This implies that $(D(\eta, f), \hat{D}(\eta, f))$ is a cone pair. □

Note that the condition “$\partial M = \emptyset$” cannot be dispensed in order to obtain the last result. In fact, let $Q^n$ be a contractible PL $n$-manifold whose boundary $\partial Q^n$ is a homology $(n - 1)$-sphere$^2$ not simply connected. Such examples are known to exist for $n \geq 5$ (see [4]). Now let $M^n$ be the homology $n$-sphere defined by

$$M^n = (v \ast \partial Q^n) \cup Q^n$$

and let $K$ and $L$ be the homology $(n + 1)$-manifolds

$$K = c \ast M^n, \quad L = v \ast Q^n.$$

One can see that the simplicial map defined on the set of vertices of $K$ by setting: $f(v_i) = v_i$, if $v_i \neq c$, $f(c) = v$ is cone-dual (see [5]). On the other hand, $K$ is a homology manifold with collared boundary, hence $K$ is an $\mathcal{L}$-manifold with boundary $(\mathcal{L}_n = $homology $(n - 1)$-spheres, $n \geq 0)$, while $L$ is not an $\mathcal{L}$-manifold.

Note that $f(\partial K) = \partial L$ is not collared in $L$.

From the above arguments, it seems natural to ask if the Theorem 3.1 can be extended to the case of an $\mathcal{L}$-manifold with boundary nonempty as soon as $f: M \to N$ satisfies the extra condition “$f(\partial M)$ is collared in $N$”.

The next theorem gives an affirmative answer to this question. It will follow that there are not cone-dual maps which preserve $\mathcal{L}$-manifold’s structure, but do not preserve the PL-homeomorphism’s class.

**THEOREM 3.2.** Let $f: M \to N$ be a surjective cone-dual map, where $M$ is an $\mathcal{L}$-manifold with boundary $\partial M$. If $f(\partial M)$ is collared in $N$, then $f$ is approximable by a PL-homeomorphism.

**PROOF.** Let $h = \dim M$. From the hypothesis and the previous theorem it follows that $N' = f(\partial M)$ is an $\mathcal{L}_{h-1}$-manifold collared in $N$. Consequently $N$ also has dimension $h$.

$^2$By a homology $n$-sphere we mean a homology $n$-manifold which has the same homology as an $n$-sphere.
Now we will prove that \((D(\tau, f), \hat{D}(\tau, f))\) is PL-homeomorphic to \((D(\tau, N), \hat{D}(\tau, N))\) for each simplex \(\tau\) of \(N\), proceeding by decreasing induction on dimension of the simplexes of \(N\).

Let \(\tau\) be an \(h\)-simplex of \(N\). Then \(D(\tau, N) = b(\tau)\) and \(\hat{D}(\tau, N) = \emptyset = \hat{D}(\tau, f)\). Because \(D(\tau, f/\partial M) = \emptyset\), \(D(\tau, f)\) is just one point. This implies that \(D(\tau, f)\) is PL-homeomorphic to \(D(\tau, N)\).

Suppose now that \(\tau\) is an \((h - 1)\)-simplex of \(N\). We will prove that there exists a PL-homeomorphism

\[
\varphi_\tau : (D(\tau, f), \hat{D}(\tau, f)) \rightarrow (D(\tau, N), \hat{D}(\tau, N))
\]

so that the following conditions hold:

(i) \(\sigma < \tau \Rightarrow \varphi_\tau(D(\sigma, f)) = \varphi_\sigma\),

(ii) \(\tau \in N' \Rightarrow \varphi_\tau(D(\tau, f/\partial M)) = D(\tau, N')\).

There are two cases according to whether \(\tau \in N'\), or not.

Case I. Assume \(\tau \notin N' = f(\partial M)\).

From above and Proposition 1.2, we have

\[
D(\tau, f) = \bigcup_{\sigma < \tau} D(\sigma, f) \cup_{\text{PL}} D(\sigma, N) = D(\tau, N).
\]

Since in this case \(D(\tau, f)\) and \(D(\tau, N)\) are cones on \(\hat{D}(\tau, f)\) and \(\hat{D}(\tau, N)\) respectively, one can extend \(\varphi_\tau\) conewise to obtain the desired PL-homeomorphism.

Case II. Assume \(\tau \in N'\).

By hypothesis there exists one only \(h\)-simplex \(\sigma\) of \(N\) such that \(\tau < \sigma\). Then we have: \(\hat{D}(\tau, N) = D(\sigma, N) = b(\sigma)\). This implies that \(\hat{D}(\tau, f) = f^{-1}(D(\sigma, N)) = D(\sigma, f)\). Hence \(\hat{D}(\tau, f)\) is a single point. On the other hand \(D(\tau, f/\partial M)\) is also a single point. It follows that \(D(\tau, f)\) is a cone on two points. Being \(D(\tau, N)\) a cone over a point, it is trivial to construct the required \(\varphi_\tau\).

Now we suppose that for all simplexes \(\tau\) of \(N\) of dimension greater than \(d\) there exists a PL-homeomorphism

\[
\varphi_\tau : (D(\tau, f), \hat{D}(\tau, f)) \rightarrow (D(\tau, N), \hat{D}(\tau, N))
\]

satisfying (i) and (ii). Let \(\tau\) be a \(d\)-simplex of \(N\).

As above we distinguish two cases. If \(\tau \notin N'\), it follows

\[
\partial D(\tau, f) = \hat{D}(\tau, f) = \bigcup_{\sigma < \tau} D(\sigma, f) \cup_{\text{PL}} D(\sigma, N) = \hat{D}(\tau, N)
\]

where \(\Phi\) is the PL-homeomorphism obtained by gluing the PL-homeomorphisms \(\varphi_\sigma\) as stated by inductive hypothesis. The required PL-homeomorphism is obtained by conical extension of \(\Phi\).

If instead \(\tau \in N'\) we see that

\[
\hat{D}(\tau, f) = \bigcup_{\sigma < \tau} D(\sigma, f) \cup_{\text{PL}} D(\sigma, N) = \hat{D}(\tau, N).
\]

Then (ii) implies that

\[
\Phi(\partial D(\tau, f/\partial M)) = \Phi \left( \bigcup_{\tau \notin N'} D(\tau, f/\partial M) \right) = \bigcup_{\tau \notin N'} D(\tau, N') = \hat{D}(\tau, N).
\]
By extending \( \Phi/\hat{D}(\tau, f/\partial M) \) conewise, we obtain a PL-homeomorphism \( \Phi': D(\tau, f/\partial M) \rightarrow D(\tau, N') \). Now we denote by \( \Phi \) the PL-homeomorphism obtained by gluing \( \Phi \) and \( \Phi' \). Because \( N' = f(\partial M) \) is collared in \( N \), we have that \( \hat{D}(\tau, N) = c\hat{D}(\tau, N') \). Since \( D(\tau, N') \) also is a cone on \( \hat{D}(\tau, N') \), then we have

\[
\hat{D}(\tau, N) \cup D(\tau, N') \cong \Sigma \hat{D}(\tau, N') = \Sigma \partial D(\tau, N').
\]

Thus it follows

\[
D(\tau, N) = c\hat{D}(\tau, N) \cong cc \partial D(\tau, N') = \Sigma \partial D(\tau, N').
\]

The desired PL-homeomorphism between \( D(\tau, f) \) and \( D(\tau, N) \) can be obtained by extending \( \Phi \) conewise.

Thus we have proved that \( (D(\tau, f), \hat{D}(\tau, f)) \) is a cone pair for each simplex \( \tau \) of \( N \). Applying 1.4 and Theorem 1.6, the assertion follows.  

4. A counterexample. In the previous section we have seen that the property of being cone-dual for a map defined on an \( \mathcal{L} \)-manifold with boundary does not suffice by itself to preserve the PL-homeomorphism’s class. It is natural to ask if a cone-dual map preserves at least the topological homeomorphism’s class.

The following example shows this to be false.

Let \( Q \) be the Mazur homology 3-sphere. Assume \( v \) a vertex of \( Q \), and denote by \( P \) the PL-manifold obtained from \( Q \) by removing the open star of \( v \). Let \( K = c*Q \) and \( L = v*P \).

We see that \( K \) and \( L \) are not homeomorphic.

In fact, because the suspension of \( Q \) is not homeomorphic to the 4-sphere, \( K-\partial K \) is not a topological manifold, and \( c \) is a singular point. On the other hand, clearly, \( L-\partial L \) is a PL-manifold. Hence a possible homeomorphism of \( K \) in \( L \) will take the point \( c \) in a point of \( \partial L \). But this is excluded by local homology’s arguments.

Finally we can easily observe that the simplicial map \( \varphi \) of \( K \) in \( L \) defined on the set of vertices of \( K \) by setting: \( \varphi(c) = v \) and \( \varphi(v_i) = v_i \) if \( v_i \neq c \), is cone-dual.

REFERENCES

5. S. Dragotti and G. Magro, Strutture simpliciali e funzioni dual coniche, Ricerche Mat. 35 (1986), 269–278.