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$\mathscr{L}$-MANIFOLDS AND CONE-DUAL MAPS

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ABSTRACT. Let $f\colon P \to Q$ be a simplicial map such that $D(\alpha, f)$, the dual to α with respect to f, is a cone, for each simplex α of Q. It is shown that if P is an $\mathscr{L}$-manifold then f is approximable by PL-homeomorphisms, provided that f satisfies an extra condition on the boundary of P.

Introduction. An interesting area of research has been that of trying to identify those maps which are approximable by PL-homeomorphisms or topological homeomorphisms. The domain and the range of these maps are spaces with extra structures, as PL-manifolds, homology manifolds, polyhedra etc. So, for example, a cellular map $f\colon M \to N$ between topological n-manifolds is approximable by homeomorphisms (see Siebenmann [**11**], for $n \neq 4, 5$. The referee pointed out to us that Freedman and Edwards have proved the result for $n = 4$ and $n = 5$, respectively).

A generalization of this result to homology manifolds by introducing a more general concept of cellularity can be found in Henderson, [**7**].

If M is a PL-manifold, a cellular map, or PL-cellular map, it is not approximable by PL-homeomorphisms. A class of maps (transversely cellular maps) which do this is given by M. Cohen in [**3**].

In a recent work, [**5**], we have studied the problem when the domain is a homology manifold, and we have found a class of maps, called cone-dual maps, preserving homology manifold's structure, but nonpreserving the PL-homeomorphism's class. In the attempt to increase the last definition in order to obtain the required approximability, we arrive at defining the strong cone-dual maps. These last maps are approximable by PL-homeomorphisms when even the domain is a simple polyhedron.

In the present work we investigate the approximability by a PL-homeomorphism or top-homeomorphism, of maps between $\mathscr{L}$-manifolds. The $\mathscr{L}$-manifolds are a class of polyhedra which includes homology manifolds without boundary or with collared boundary.

The results obtained may be summarized as follows:

(1) A cone-dual map $f\colon M \to N$ is approximable by a PL-homeomorphism where M is an $\mathscr{L}$-manifold without boundary (Theorem 3.1).

(2) If M is an $\mathscr{L}$-manifold with boundary ∂M, a cone-dual map $f\colon M \to N$ is approximable by a PL-homeomorphism provided $f(\partial M)$ is collared in N (Theorem 3.2).

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(3) The condition "$f(\partial M)$ is collared in N" is necessary to approximate f to a top-homeomorphism, i.e. there exist cone-dual maps $f\colon M \to N$ where M is an $\mathscr{L}$-manifold with boundary and N is not topological homeomorphic to M (§4).

1. Cone-dual maps. Let K be a simplicial complex, for each simplex α of K, the dual to α in K, denoted $D(\alpha, K)$, and its subcomplex $\dot{D}(\alpha, K)$ are defined by

$$D(\alpha, K) = \{b(\sigma_1)\cdots b(\sigma_h) | \alpha < \sigma_1 < \cdots < \sigma_h < K\},$$

and

$$\dot{D}(\alpha, K) = \{b(\sigma_1)\cdots b(\sigma_h) | \alpha \lneqq \sigma_1 < \cdots < \sigma_h < K\}$$

where $b(\sigma_i)$ denotes the barycenter of σ_i.

It is known that

(a) $D(\alpha, K) = b(\alpha) * \dot{D}(\alpha, K)$

(b) $\dot{D}(\alpha, K) \cong_{\mathrm{PL}} \mathrm{Lk}(\alpha, K)$

Throughout this paper all polyhedra are assumed to be compact and connected.

Let $f\colon K \to L$ be a simplicial map, L' the first barycentric subdivision of L, and K' a barycentric subdivision of K chosen so that f is also simplicial with respect to K' and L'. For each simplex α of L, $D(\alpha, f)$ and $\dot{D}(\alpha, f)$ are the subcomplexes of K' defined by

$$D(\alpha, f) = \{b(\sigma_1)\cdots b(\sigma_h) | \alpha < f(\sigma_1),\ \sigma_1 < \cdots < \sigma_h < K\},$$

and

$$\dot{D}(\alpha, f) = \{b(\sigma_1)\cdots b(\sigma_h) | \alpha \lneqq f(\sigma_1),\ \sigma_1 < \cdots < \sigma_h < K\}.$$

$D(\alpha, f)$ is called the dual to α with respect to f.

We refer to [**2**, **3**] for the proofs of following results.

PROPOSITION 1.1. $D(\alpha, f) = f^{-1}(D(\alpha, L))$; $\dot{D}(\alpha, f) = f^{-1}(\dot{D}(\alpha, L))$.

PROPOSITION 1.2. *K is the union of the duals of the simplexes of L with respect to f. Moreover we have:*

(a) $\dot{D}(\alpha, f) = \bigcup_{\alpha < \dot{\beta}} D(\beta, f)$,

(b) $D(\alpha, f) \cap D(\beta, f) = D(\alpha \cdot \beta, f)$,

where $\alpha \cdot \beta$ is the simplex spanned by α and β if there is one, $\alpha \cdot \beta = \varnothing$ otherwise, and $D(\varnothing, f) = \varnothing$.

PROPOSITION 1.3. *If $\alpha^{i-1} < \alpha^i$, then $D(\alpha^i, f)$ is a regular neighbourhood of $f^{-1}(b(\alpha^i))$ in $\dot{D}(\alpha^{i-1}, f)$ with boundary $\dot{D}(\alpha^i, f)$. (For $i = 0$ we assume $\dot{D}(\alpha^{-1}, f) = K'$.)*

We refer to [**5**] for the following definition and proposition.

A simplicial map $f\colon K \to L$ is called strong cone-dual if $(D(\alpha, f), \dot{D}(\alpha, f))$ is a cone pair for each simplex α of L, i.e. there is a PL-homeomorphism of $D(\alpha, f)$ onto a cone on $\dot{D}(\alpha, f)$, which maps $\dot{D}(\alpha, f)$ on $\dot{D}(\alpha, f)$ (see Stallings [**12**]).

PROPOSITION 1.4. *If $f\colon K \to L$ is a surjective strong cone-dual map, then there is a PL-homeomorphism $\hat{f}$ of K into L such that $\hat{f}(D(\alpha, f)) = D(\alpha, L)$, for each simplex α of L.*

In [**5**] we have defined cone-dual maps between homology manifolds. It is possible to extend this definition to maps between polyhedra as soon as we define the boundary of a polyhedron.

Given an h-polyhedron P, the boundary of P is the subpolyhedron defined inductively by

$$\partial P = \begin{cases} \varnothing, & h = 0, \\ \{x \in P | \mathrm{Lk}(x, P) = \text{ point or } \partial \mathrm{Lk}(x, P) \neq \varnothing\}, & h \geq 1,^1 \end{cases}$$

DEFINITION 1.5. *A simplicial map $f\colon P \to Q$ is called cone-dual with respect to the triangulations K and L of the polyhedra P and Q if $D(\sigma, f)$ is a cone for each simplex σ of $f(K)$, and $D(\sigma, f/\partial K)$ is a cone for each simplex σ of $f(\partial K)$.*

The next theorem shows that Definition 1.5 does not depend on the triangulations chosen.

THEOREM 1.6. *Let $f\colon K \to L$ be a cone-dual map, $\overline{K}$ and $\overline{L}$ triangulations of K and L such that f is also simplicial. Then f is cone-dual with respect to $\overline{K}$ and $\overline{L}$.*

To prove this theorem we will use the following lemma.

LEMMA 1.7. *Let $f\colon K \to L$ be a cone-dual map, if β^i is an i-simplex of $f(K)$ and β^j is a j-face of β^i, then a regular nbd of $f^{-1}(b(\beta^i))$ in $\dot{D}(\beta^j, f)$ is PL-homeomorphic to a cone. (If $\beta^j = \varnothing$ assume $\dot{D}(\beta^j, f) = K'$.)*

PROOF. If $j = i - 1$, the result follows from Proposition 1.3. So we suppose $j < i - 1$.

Let $\beta^{j+1}, \beta^{j+2}, \ldots, \beta^{i-1}$ be a finite sequence of simplexes of $f(K)$ such that $\beta^j < \beta^{j+1} < \cdots < \beta^{i-1} < \beta^i$. By Proposition 1.3, $D(\beta^h, f)$ is a regular nbd of $f^{-1}(b(\beta^h))$ in $\dot{D}(\beta^{h-1}, f)$ with boundary $\dot{D}(\beta^h, f)$. Hence $\dot{D}(\beta^h, f)$ is bicollared in $\dot{D}(\beta^{h-1}, f)$. This implies that a regular nbd U of $\dot{D}(\beta^{i-1}, f)$ in $\dot{D}(\beta^j, f)$ is PL-homeomorphic to $\dot{D}(\beta^{i-1}, f) \times [-1, 1]^r$ ($r = i - j - 1$), by a PL-homeomorphism

$$\varphi\colon \dot{D}(\beta^{i-1}, f) \times [-1, 1]^r \to U$$

so that $\varphi(\dot{D}(\beta^{i-1}, f) \times \{0\}^r) = \dot{D}(\beta^{i-1}, f)$. Since $D(\beta^i, f)$ is a a regular nbd of $f^{-1}(b(\beta^i))$ in $\dot{D}(\beta^{i-1}, f)$, a regular nbd of $f^{-1}(b(\beta^i))$ in $\dot{D}(\beta^j, f)$ is PL-homeomorphic to $D(\beta^i, f) \times [-1, 1]^r$, which is a cone. $\square$

PROOF OF THEOREM 1.6. First suppose that $\overline{L}$ is obtained from L by starring at only point $v = b(\tilde{v})$. Generally for each simplex α of $\overline{L}$, by $\tilde{\alpha}$ we mean the carrier of α in L. Moreover we denote by $\overline{D}(\alpha, f)$ and by $\dot{\overline{D}}(\alpha, f)$ the corresponding carrier of $D(\alpha, f)$ and $\dot{D}(\alpha, f)$ with respect to $f\colon \overline{K} \to \overline{L}$, i.e.: $\overline{D}(\alpha, f) = f^{-1}(D(\alpha, \overline{L})$, $\dot{\overline{D}}(\alpha, f)) = f^{-1}(\dot{D}(\alpha, \overline{L}))$.

To prove that $\overline{D}(\alpha, f)$ is a cone for each α of $\overline{L}$, we proceed to consider the various cases.

Case I. Assume $\alpha \in \overline{L} \cap L$.

Note that if α does not lie in $\mathrm{Lk}(v, \overline{L})$, we have $D(\alpha, \overline{L}) = D(\alpha, L)$, hence $\overline{D}(\alpha, f) = D(\alpha, f)$.

If α is a face of some simplex which contains v, then there exists a PL-homeomorphism between $\overline{D}(\alpha, f)$ and $D(\alpha, f)$. In fact let $\varphi(b(\tau)) = b(\tilde{\tau})$ for each vertex $b(\tau)$

[1] An equivalent definition of boundary can be found in [**13**].

of $\overline{D}(\alpha, f)$. Evidently, if α, in $\overline{L}$, is a face of $f(\tau)$, then α, as simplex of L, will be a face of $f(\tilde{\tau})$. Hence $b(\tilde{\tau})$ is a vertex of $D(\alpha, f)$. It follows that φ carries the vertices of $\overline{D}(\alpha, f)$ into vertices of $D(\alpha, f)$.

φ is an injective map. In fact if $\varphi(b(\tau)) = \varphi(b(\delta))$, then τ and δ are simplexes of $\overline{K}$ so that $\tilde{\tau} = \tilde{\delta}$. This implies either $\tau = \delta$, or v lies in α ($\alpha < f(\tau), \alpha < f(\delta)$). Since $\alpha \in \overline{L} \cap L$, the last eventuality does not occur.

Trivially φ is surjective, and hence bijective.

Observe that if $\tau_1 < \tau_2$, then $\tilde{\tau}_1 < \tilde{\tau}_2$. Hence if $b(\tau_1), \ldots, b(\tau_h)$ lie in a simplex of $\overline{D}(\alpha, f)$, then $b(\tilde{\tau}_1), \ldots, b(\tilde{\tau}_h)$ lie in a simplex of $D(\alpha, f)$. Therefore φ can be extended to a PL-homeomorphism, which we will denote again by φ, of $\overline{D}(\alpha, f)$ onto $D(\alpha, f)$.

Observe that, in this case, if $\alpha \lneqq f(\tau)$ then $\alpha \lneqq f(\tilde{\tau})$. Consequently φ takes $\dot{\overline{D}}(\alpha, f)$ onto $\dot{D}(\alpha, f)$, and φ coincides with identity when α does not lie in the closure of the star of v in $\overline{L}$.

Thus if $\alpha \in L \cap \overline{L}$, then $\overline{D}(\alpha, f)$, being PL-homeomorphic to the cone $D(\alpha, f)$, is a cone.

Case II. Assume $\alpha \in \overline{L} - L$, $\alpha \neq v$.

In this case v is a vertex of α. Since $\alpha \neq v$, there exists a simplex $\beta \in L \cap \overline{L}$ so that β is a 1-codimensional face of α ($\beta =$ opposite face to v). Then $\overline{D}(\alpha, f)$ is a regular nbd of $f^{-1}(b(\alpha))$ in $\dot{\overline{D}}(\beta, f)$. From the previous case it follows that $\dot{\overline{D}}(\beta, f)$ is PL-homeomorphic to $\dot{D}(\beta, f)$. On the other hand $f^{-1}(b(\alpha))$ is PL-homeomorphic to $f^{-1}(b(\tilde{\alpha}))$. Now observe that β is also a face of $\tilde{\alpha}$, but in general it is not a 1-codimensional face. However, by Lemma 1.7, we can assert that $f^{-1}(b(\tilde{\alpha}))$ has in $\dot{D}(\beta, f)$ a regular nbd which is a cone. Thus, from the uniqueness theorem for regular nbd, it follows that $\overline{D}(\alpha, f)$ is a cone.

Case III. Assume $\alpha = v$.

By Lemma 1.7, $f^{-1}(v) = f^{-1}(b(\tilde{v}))$ has a regular nbd which is a cone in K'. As above, using the uniqueness theorem for regular nbd, and the fact that K' is also a subdivision of $\overline{K}$, we have that $\overline{D}(v, f)$ is a cone.

In order to prove the result in the general case, we must show that, assuming L and $\overline{L}$ as above, we can exchange their roles with respect to f. That is, if we suppose that $\overline{D}(\alpha, f)$ is a cone for every simplex α of $\overline{L}$, then $D(\alpha, f)$ is a cone for every $\alpha \in L$.

In fact, if $\alpha \in \overline{L} \cap L$, we have proved that $D(\alpha, f)$ is PL-homeomorphic to $\overline{D}(\alpha, f)$, and hence $D(\alpha, f)$ is a cone. If instead α lies in $L - \overline{L}$, then it is the carrier of a simplex $\overline{\alpha}$ of $\overline{L}$ of the same dimension. Then, if β is a simplex of $L \cap \overline{L}$ and a 1-codimensional face of α and $\overline{\alpha}$, reasoning as before (Case II), we have that $f^{-1}(b(\alpha))$ has a regular nbd in $\dot{D}(\beta, f)$ which is a cone. Hence $D(\alpha, f)$ is a cone.

Finally, to complete the proof, it suffices to recall that equivalent triangulations of L have a common subdivision, and that every subdivision L' of L can be obtained from L by a finite number of subdivisions $L' = L_h \lhd L_{h-1} \lhd \cdots \lhd L_1 = L$ so that L_i is obtained from L_{i-1} by introducing an only vertex. $\square$

2. Duals and $\mathscr{L}$-manifolds. In this section we investigate the dual structure induced by a simplicial map $f \colon K \to L$, on K, when K is an $\mathscr{L}$-manifold.

For the reader's convenience, we reproduce here the definition of $\mathscr{L}$-manifold, according to [**1**].

Suppose we are given a class $\mathscr{L}_n$, for each $n \geq 0$, of $(n-1)$-polyhedra (closed under PL-homeomorphisms), which satisfies:

(1) Each member of $\mathscr{L}_n$ is a polyhedron whose links lie in $\mathscr{L}_{n-1}$.

(2) $\Sigma\mathscr{L}_{n-1} \subseteq \mathscr{L}_n$ (i.e. the suspension of an $(n-1)$-link is an n-link).

(3) $c\mathscr{L}_{n-1} \cap \mathscr{L}_n = \varnothing$ (i.e. the cone on $(n-1)$-link is never a link).

Then an $\mathscr{L}_n$-manifold M is a polyhedron whose links lie either in $\mathscr{L}_n$ or $c\mathscr{L}_{n-1}$. The boundary of M, ∂M, consists of points whose links lie in the latter class.

As an immediate consequence of the definition we observe that the boundary of an $\mathscr{L}_n$-manifold M is itself an $\mathscr{L}_{n-1}$-manifold ∂M without boundary. Furthermore ∂M is collared in M.

REMARK 2.1. The link of an i-simplex in an $\mathscr{L}_n$-manifold lies either in $\mathscr{L}_{n-i}$ or in $c\mathscr{L}_{n-i-1}$.

REMARK 2.2. Every polyhedron of $\mathscr{L}_n$ is an $\mathscr{L}_{n-1}$-manifold without boundary.

REMARK 2.3. An $\mathscr{L}$-manifold, which is a cone, is a cone on the complete boundary.

In fact, let $M = v * H$ be an $\mathscr{L}$-manifold. Since $H = \mathrm{Lk}(v, M)$, H lies either in $\mathscr{L}_n$ or in $c\mathscr{L}_{n-1}$. If H lies in $\mathscr{L}_n$, then $\partial H = \varnothing$. Hence H is the complete boundary of M. If instead $H = cX$, with $X \in \mathscr{L}_{n-1}$, then we have: $M = ccX \cong_{\mathrm{PL}} c\Sigma X$ and $\partial(c\Sigma X) = \Sigma X \cong_{\mathrm{PL}} \partial M$. This implies $M \cong_{\mathrm{PL}} c\partial M$.

THEOREM 2.4. *Let $f\colon M \to L$ be a simplicial map, where M is an $\mathscr{L}_n$-manifold and L a polyhedron. For each i-simplex γ of $f(M)$, $D(\gamma, f)$ is an $\mathscr{L}_{n-i}$-manifold with boundary $\dot{D}(\gamma, f) \cup D(\gamma, f/\partial M)$.*

PROOF. Let $\sigma = b(\sigma_0)\cdots b(\sigma_h)$ be a simplex of $D(\gamma, f) - \dot{D}(\gamma, f)$, we have that (see [**3**])

$$\begin{aligned}
\mathrm{Lk}(\sigma, D(\gamma, f)) &= \{b(\tau_0)\cdots b(\tau_q) | \gamma = f(\tau_0) = \cdots = f(\tau_q); \tau_q < \dot{\sigma}_0\} \\
&\quad * \dot{D}(\sigma_0, \dot{\sigma}_1) * \cdots * \dot{D}(\sigma_{h-1}, \dot{\sigma}_h) * \dot{D}(\sigma_h, M) \\
&= [(f/\dot{\sigma}_0)^{-1}(b(\gamma))] * \dot{D}(\sigma_0, \dot{\sigma}_1) * \cdots * \dot{D}(\sigma_{h-1}, \dot{\sigma}_h) * \dot{D}(\sigma_h, M).
\end{aligned}$$

Let $\dim \sigma_h = n_h$, using results about duals in a PL-manifold, it follows that $\mathrm{Lk}(\sigma, D(\gamma, f)) = \Sigma^r \mathrm{Lk}(\sigma_h, M)$, where $r = n_h - h - i$. Therefore $\mathrm{Lk}(\sigma, D(\gamma, f))$ lies either in $\mathscr{L}_{n-i-h}$ or $c\mathscr{L}_{n-i-h-1}$, depending on whether σ_h (and hence σ) is in $M - \partial M$ or not.

Suppose now that σ lies in $\dot{D}(\gamma, f)$, we have

$$\mathrm{Lk}(\sigma, D(\gamma, f)) = D(\gamma, f/\dot{\sigma}_0) * \dot{D}(\sigma_0, \dot{\sigma}_1) * \cdots * \dot{D}(\sigma_{h-1}, \dot{\sigma}_h) * \dot{D}(\sigma_h, M).$$

Since $D(\gamma, f/\dot{\sigma}_0)$ is a PL-ball (see [**3**]), it follows that

$$\mathrm{Lk}(\sigma, D(\gamma, f)) \underset{\mathrm{PL}}{\cong} v * \Sigma^{k+1} \mathrm{Lk}(\sigma_h, M), \qquad \text{where } k = n_h - i - h - 2.$$

If σ_h does not lie in ∂M, then $\mathrm{Lk}(\sigma_h, M)$ lies in $\mathscr{L}_{n-n_h-1}$ and hence $\mathrm{Lk}(\sigma_h, D(\gamma, f))$ lies in $c\mathscr{L}_{n-i-h-1}$. If σ_h lies in ∂M, then $\mathrm{Lk}(\sigma_h, M)$ is a cone on a polyhedron X of $\mathscr{L}_{n-n_h-1}$, it follows that

$$\mathrm{Lk}(\sigma, D(\gamma, f)) \underset{\mathrm{PL}}{\cong} v * c\Sigma^{k+1} X \underset{\mathrm{PL}}{\cong} c\Sigma^{k+2} X.$$

So we have that $D(\gamma, f)$ is an $\mathscr{L}_{n-i}$-manifold.

Moreover $\mathrm{Lk}(\sigma, D(\gamma, f))$ is a cone if and only if σ lies either in ∂M or in $\dot{D}(\gamma, f)$. This implies that

$$\partial D(\gamma, f) = \dot{D}(\gamma, f) \cup D(\gamma, f/\partial M). \quad \square$$

3. Cone-dual maps and $\mathscr{L}$-manifolds. Let $f\colon M \to N$ be a cone-dual map. In [**5**] we have proved that if M is a homology manifold, then $f(M)$ is itself a homology manifold. But in general $f(M)$ is not PL-homeomorphic to M.

The next theorem shows that this result can be improved if we suppose that M is a homology manifold, or, more generally an $\mathscr{L}$-manifold, without boundary.

THEOREM 3.1. *Let $f\colon M \to N$ be a surjective cone-dual map. If M is an $\mathscr{L}$-manifold without boundary, then F is approximable by a PL-homeomorphism.*

PROOF. By Proposition 1.4 and Theorem 1.6 it suffices to prove that f is a strong cone-dual map.

Let η be a simplex of N. By hypothesis $D(\eta, f)$ is PL-homeomorphic to a cone cX. On the other hand, from Theorem 2.4, $D(\eta, f)$ is an $\mathscr{L}$-manifold with boundary $\dot{D}(\eta, f)$. Hence, using Remark 2.3, we can suppose $X = \dot{D}(\eta, f)$. This implies that $(D(\eta, f), \dot{D}(\eta, f))$ is a cone pair. $\square$

Note that the condition "$\partial M = \varnothing$" cannot be dispensed in order to obtain the last result. In fact, let Q^n be a contractible PL n-manifold whose boundary ∂Q^n is a homology $(n-1)$-sphere[2] not simply connected. Such examples are known to exist for $n \geq 5$ (see [**4**]). Now let M^n be the homology n-sphere defined by

$$M^n = (v * \partial Q^n) \cup Q^n$$

and let K and L be the homology $(n+1)$-manifolds

$$K = c * M^n, \quad L = v * Q^n.$$

One can see that the simplicial map defined on the set of vertices of K by setting: $f(v_i) = v_i$, if $v_i \neq c$, $f(c) = v$ is cone-dual (see [**5**]). On the other hand, K is a homology manifold with collared boundary, hence K is an $\mathscr{L}$-manifold with boundary ($\mathscr{L}_n$ =homology $(n-1)$-spheres, $n \geq 0$), while L is not an $\mathscr{L}$-manifold.

Note that $f(\partial K) = \partial L$ is not collared in L.

From the above arguments, it seems natural to ask if the Theorem 3.1 can be extended to the case of an $\mathscr{L}$-manifold with boundary nonempty as soon as $f\colon M \to N$ satisfies the extra condition "$f(\partial M)$ is collared in N".

The next theorem gives an affirmative answer to this question. It will follow that there are not cone-dual maps which preserve $\mathscr{L}$-manifold's structure, but do not preserve the PL-homeomorphism's class.

THEOREM 3.2. *Let $f\colon M \to N$ be a surjective cone-dual map, where M is an $\mathscr{L}$-manifold with boundary ∂M. If $f(\partial M)$ is collared in N, then f is approximable by a PL-homeomorphism.*

PROOF. Let $h = \dim M$. From the hypothesis and the previous theorem it follows that $N' = f(\partial M)$ is an $\mathscr{L}_{h-1}$-manifold collared in N. Consequently N also has dimension h.

[2] By a homology n-sphere we mean a homology n-manifold which has the same homology as an n-sphere.

Now we will prove that $(D(\tau, f),\ \dot{D}(\tau, f))$ is PL-homeomorphic to $(D(\tau, N),\ \dot{D}(\tau, N))$ for each simplex τ of N, proceeding by decreasing induction on dimension of the simplexes of N.

Let τ be an h-simplex of N. Then $D(\tau, N) = b(\tau)$ and $\dot{D}(\tau, N) = \varnothing = \dot{D}(\tau, f)$. Because $D(\tau, f/\partial M) = \varnothing$, $D(\tau, f)$ is just one point. This implies that $D(\tau, f)$ is PL-homeomorphic to $D(\tau, N)$.

Suppose now that τ is an $(h-1)$-simplex of N. We will prove that there exists a PL-homeomorphism

$$\varphi_\tau\colon (D(\tau, f), \dot{D}(\tau, f)) \to (D(\tau, N), \dot{D}(\tau, N))$$

so that the following conditions hold:

(i) $\sigma < \tau \Rightarrow \varphi_\tau / D(\sigma, f) = \varphi_\sigma$,

(ii) $\tau \in N' \Rightarrow \varphi_\tau(D(\tau, f/\partial M)) = D(\tau, N')$.

There are two cases according to whether $\tau \in N'$, or not.

Case I. Assume $\tau \notin N' = f(\partial M)$.

From above and Proposition 1.2, we have

$$\dot{D}(\tau, f) = \bigcup_{\tau < \dot{\sigma}^h} D(\sigma^h, f) \overset{\varphi}{\underset{\text{PL}}{\cong}} \bigcup_{\tau < \dot{\sigma}^h} D(\sigma^h, N) = \dot{D}(\tau, N).$$

Since in this case $D(\tau, f)$ and $D(\tau, N)$ are cones on $\dot{D}(\tau, f)$ and $\dot{D}(\tau, N)$ respectively, one can extend φ conewise to obtain the desired PL-homeomorphism.

Case II. Assume $\tau \in N'$.

By hypothesis there exists one only h-simplex σ^h of N such that $\tau < \sigma^h$. Then we have: $\dot{D}(\tau, N) = D(\sigma^h, N) = b(\sigma^h)$. This implies that: $\dot{D}(\tau, f) = f^{-1}(D(\sigma^h, N)) = D(\sigma^h, f)$. Hence $\dot{D}(\tau, f)$ is a single point. On the other hand $D(\tau, f/\partial M)$ is also a single point. It follows that $D(\tau, f)$ is a cone on two points. Being $D(\tau, N)$ a cone over a point, it is trivial to construct the required φ_τ.

Now we suppose that for all simplexes τ of N of dimension greater than d there exists a PL-homeomorphism

$$\varphi_\tau\colon (D(\tau, f), \dot{D}(\tau, f)) \to (D(\tau, N), \dot{D}(\tau, N))$$

satisfying (i) and (ii). Let τ be a d-simplex of N.

As above we distinguish two cases. If $\tau \notin N'$, it follows

$$\partial D(\tau, f) = \dot{D}(\tau, f) = \bigcup_{\tau < \dot{\sigma}} D(\sigma, f) \overset{\Phi}{\underset{\text{PL}}{\cong}} \bigcup_{\tau < \dot{\sigma}} D(\sigma, N) = \dot{D}(\tau, N)$$

where Φ is the PL-homeomorphism obtained by gluing the PL-homeomorphisms φ_σ as stated by inductive hypothesis. The required PL-homeomorphism is obtained by conical extension of Φ.

If instead $\tau \in N'$ we see that

$$\dot{D}(\tau, f) = \bigcup_{\tau < \dot{\sigma}} D(\sigma, f) \overset{\Phi}{\underset{\text{PL}}{\cong}} \bigcup_{\tau < \dot{\sigma}} D(\sigma, N) = \dot{D}(\tau, N).$$

Then (ii) implies that

$$\Phi(\partial D(\tau, f/\partial M)) = \Phi\left(\bigcup D(\sigma, f/\partial M)\right) = \bigcup D(\sigma, N') = \dot{D}(\tau, N).$$

By extending $\Phi/\dot{D}(\tau, f/\partial M)$ conewise, we obtain a PL-homeomorphism $\Phi'\colon D(\tau, f/\partial M) \to D(\tau, N')$. Now we denote by $\overline{\Phi}$ the PL-homeomorphism obtained by gluing Φ and Φ'. Because $N' = f(\partial M)$ is collared in N, we have that $\dot{D}(\tau, N) = c\dot{D}(\tau, N')$. Since $D(\tau, N')$ also is a cone on $\dot{D}(\tau, N')$, then we have

$$\dot{D}(\tau, N) \cup D(\tau, N') \underset{\mathrm{PL}}{\cong} \Sigma\dot{D}(\tau, N') = \Sigma\partial D(\tau, N').$$

Thus it follows

$$\begin{aligned} D(\tau, N) = c\dot{D}(\tau, N) &\underset{\mathrm{PL}}{\cong} cc\partial D(\tau, N') \\ &\underset{\mathrm{PL}}{\cong} c\Sigma\partial D(\tau, N') \underset{\mathrm{PL}}{\cong} c * (\dot{D}(\tau, N) \cup D(\tau, N')). \end{aligned}$$

The desired PL-homeomorphism between $D(\tau, f)$ and $D(\tau, N)$ can be obtained by extending $\overline{\Phi}$ conewise.

Thus we have proved that $(D(\tau, f),\ \dot{D}(\tau, f))$ is a cone pair for each simplex τ of N. Applying 1.4 and Theorem 1.6, the assertion follows. □

4. A counterexample. In the previous section we have seen that the property of being cone-dual for a map defined on an $\mathscr{L}$-manifold with boundary does not suffice by itself to preserve the PL-homeomorphism's class. It is natural to ask if a cone-dual map preserves at least the topological homeomorphism's class.

The following example shows this to be false.

Let Q be the Mazur homology 3-sphere. Assume v a vertex of Q, and denote by P the PL-manifold obtained from Q by removing the open star of v. Let $K = c * Q$ and $L = v * P$.

We see that K and L are not homeomorphic.

In fact, because the suspension of Q is not homeomorphic to the 4-sphere, $K - \partial K$ is not a topological manifold, and c is a singular point. On the other hand, clearly, $L - \partial L$ is a PL-manifold. Hence a possible homeomorphism of K in L will take the point c in a point of ∂L. But this is excluded by local homology's arguments.

Finally we can easily observe that the simplicial map φ of K in L defined on the set of vertices of K by setting: $\varphi(c) = v$ and $\varphi(v_i) = v_i$ if $v_i \neq c$, is cone-dual.

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