PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 4, August 1988

UNIVERSAL SPACES FOR COUNTABLE DIMENSIONAL METRIC SPACES

TATSUO GOTO

(Communicated by Dennis Burke)

Dedicated to Yukihiro Kodama on his 60th birthday

ABSTRACT. Let $H(A)$ be the Dowker's generalized Hilbert space with weight $|A|$, where A is any infinite set, and $H_\infty(A)$ its subspace consisting of all points which have only finitely many rational coordinates distinct from zero. Using a result of E. Pol, it will be shown that $H_\infty(A)$ is a universal space for countable dimensional metric spaces with weight $\leq |A|$.

1. Introduction. A metrizable space X is called countable dimensional if it can be represented as the union of a sequence $X_1, X_2, \dots$ of subspaces such that $\dim X_i \leq 0$, $i = 1, 2, \dots$, where dim denotes the covering dimension. Universal spaces for countable dimensional metric spaces were studied by J. Nagata and it was shown that

(a) the subspace N^ω of the Hilbert cube I^ω consisting of all points which have at most finitely many rational coordinates is a universal space for countable dimensional separable metric spaces [**3**, Corollary 4.4], and

(b) the subspace $K_\infty(A)$ of the Cartesian product $P(A)$ of $\aleph_0$ copies of the star space $S(A)$ is a universal space for countable dimensional metric space with weight $\leq |A|$, where $|A|$ denotes the cardinality of A and $K_\infty(A)$ consists of all points in $P(A)$ which have only finitely many rational coordinates distinct from zero [**4**, Theorem 2].

On the other hand, the generalized Hilbert space $H(A)$ was shown to be universal for all metrizable space with weight $\leq |A|$ by C. H. Dowker [**1**, Lemma 1]. The purpose of this note is to show the following result.

THEOREM. *Let $H_\infty(A)$ be the set consisting of all points in $H(A)$ at most finitely many of whose coordinates are rational different from zero. Then $H_\infty(A)$ is a universal space for countable dimensional metric spaces with weight $\leq |A|$.*

2. Proof of Theorem. Let A be any infinite set and R the set of real numbers. Then the Hilbert space $H = H(A)$ can be defined as a space of all points x in R^A such that $\sum_\alpha x(\alpha)^2 < \infty$, and its norm is defined by $||x|| = (\sum_\alpha x(\alpha)^2)^{1/2}$. We denote by $B_\delta(x)$ the spherical neighborhood of x with radius δ, i.e. $B_\delta(x) = \{x' \in H\colon d(x, x') < \delta\}$ where d is the metric defined by $d(x, x') = ||x - x'||$. For every integer $n \geq 0$, let $H_n = H_n(A)$ be the set of points in H exactly n of whose coordinates are nonzero rationals. Thus the space H_0 consists of all points in H whose nonzero coordinates are irrational, and E. Pol established the equality

Received by the editors July 15, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F45, 54E35.
Key words and phrases. Countable dimensional, universal space, metric space, Hilbert space.

$\dim H_0 = 1$ [**4**, Example 1.6]. Using this result we now prove

LEMMA 1. *$\dim H_n = 1$ for every integer $n \geq 0$.*

PROOF. First we note that

(1) if J is any interval in the real line R such that $0 \notin \bar{J}$ (= the closure of J) and x is a point in H, then the set $\{\alpha \in A\colon x(\alpha) \in J\}$ is finite.

Now we assume that A is well ordered with an ordering relation $<$, and denote by A_n, $n \geq 1$, the family of all subsets of A which consist of exactly n elements. Thus every ξ in A_n can be uniquely expressed as $\xi = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Let Q_0 be the set of nonzero rationals and $P_0 = R \backslash Q_0$, i.e. the set of irrationals and zero. For each $\rho = (r_1, \ldots, r_n) \in Q_0^n$ and $\xi = (\alpha_1, \ldots, \alpha_n) \in A_n$, we define

$$H(\rho, \xi) = \{x \in H_n\colon x(\alpha_i) = r_i,\ i = 1, \ldots, n\}.$$

Note that $x(\alpha) \in P_0$ for each $x \in H(\rho, \xi)$ and $\alpha \in A \backslash \xi$, and it is clear that

(2) $H(\rho, \xi)$ is closed in H_n, and

(3) $\dim H(\rho, \xi) = 1$

because $H(\rho, \xi)$ is homeomorphic to $H_0(A \backslash \xi)$ which has covering dimension one by the result of E. Pol cited above. Now let us put $H(\rho) = \bigcup\{H(\rho, \xi);\ \xi \in A_n\}$. Clearly the collection $\{H(\rho, \xi)\colon \xi \in A_n\}$ is pairwise disjoint. Moreover we prove

(4) $\{H(\rho, \xi)\colon \xi \in A_n\}$ is a discrete collection in $H(\rho)$. Indeed, if $x \in H(\rho)$ and $\rho = (r_1, \ldots, r_n)$, then $x \in H(\rho, \xi)$ for some $\xi = (\alpha_1, \ldots, \alpha_n) \in A_n$, i.e. $x(\alpha_i) = r_i$ for $i = 1, \ldots, n$. Applying (1), we can take a $\delta > 0$ such that

(5) $\delta < |x(\alpha) - r_i|$ for each $\alpha \in A \backslash \xi$ and $i = 1, \ldots, n$ because $x(\alpha) \in P_0$ for each $\alpha \in A \backslash \xi$. Let x' be a point of $B_\delta(x) \cap H(\rho)$; then there exists $\xi' = (\alpha_1', \ldots, \alpha_n') \in A_n$ such that $x' \in H(\rho, \xi')$, and hence $x'(\alpha_i') = r_i$ for every i. Since $|x(\alpha) - x'(\alpha)| \leq d(x, x') < \delta$, it follows from (5) that $x'(\alpha) \notin \{r_1, \ldots, r_n\}$ and hence $x'(\alpha) \in P_0$ for every $\alpha \in A \backslash \xi$. Therefore we have $\xi' = \xi$ and $B_\delta(x) \cap H(\rho) \subseteq H(\rho, \xi)$, which proves (4). From (2), (3) and (4), we obtain

(6) $\dim H(\rho) = 1$ for every $\rho \in Q_0^n$.

Since Q_0^n is countable, to prove Lemma 1 it suffices to show

(7) $H(\rho)$ is closed in H_n for every ρ.

Let x be a point in $H_n \backslash H(\rho)$ and $\rho = (r_1, \ldots, r_n)$; then there exist $\xi = (\alpha_1, \ldots, \alpha_n) \in A_n$ and $\sigma = (s_1, \ldots, s_n) \in Q_0^n$ such that $x \in H(\sigma, \xi)$. As $x \notin H(\rho)$, we have $\rho \neq \sigma$ and $r_k \neq s_k$ for some k. Applying (1) again, we can choose a $\delta > 0$ satisfying

(8) $\delta < |r_k - s_k|$, and

(9) $\delta < |r_i - x(\alpha)|$ for each i and $\alpha \in A \backslash \xi$.

To prove $B_\delta(x) \cap H(\rho) = \varnothing$, assume the contrary, i.e. $x' \in B_\delta(x) \cap H(\rho)$. Since $d(x, x') < \delta$, it follows from (9) that $x'(\alpha) \notin \{r_1, \ldots, r_n\}$ and hence $x'(\alpha) \in P_0$ for every $\alpha \in A \backslash \xi$, so that $x'(\alpha_i) = r_i$, $i = 1, \ldots, n$. Then we have $|r_k - s_k| = |x'(\alpha_k) - x(\alpha_k)| \leq d(x, x') < \delta$ which contradicts (8), and hence (7) is proved. This completes the proof of Lemma 1.

Since the space $H_\infty(A)$ can be represented as the union of subspaces $H_n = H_n(A)$, $n = 1, 2, \ldots$, we see that $H_\infty(A)$ is countable dimensional by virtue of Lemma 1 and the weight of $H_\infty(A)$ does not exceed the weight of $H(A) = |A|$. Therefore to prove Theorem it suffices to show the following Lemma 2, since the space $K_\infty(A)$, as cited in 1(b), is a universal space for countable dimensional metric spaces with weight $\leq |A|$.

LEMMA 2. *The space $K_\infty(A)$ can be topologically embedded as a subspace of $H_\infty(A)$.*

PROOF. Let $S'(A)$ be the subspace of $H(A)$ defined by

$$S'(A) = \{x \in H(A)\colon |s(x)| \leq 1 \text{ and } 0 < x(\alpha) \leq 1 \text{ if } \alpha \in s(x)\},$$

where $s(x) = \{\alpha \in A\colon x(\alpha) \neq 0\}$. It is easy to see that $S'(A)$ is homeomorphic to the star space $S(A)$ defined in [**5**, p. 184]. Since A is infinite we can choose a sequence $\{A_i\}$ of pairwise disjoint sets such that $A = \bigcup A_i$ and $|A_i| = |A|$ for all i. Then $S'(A_i)$ is also homeomorphic to the star space $S(A)$. Let

$$K'(A) = \{x \in H(A)\colon |s(x) \cap A_i| \leq 1 \text{ and } 0 < x(\alpha) \leq 1/i \quad \text{if } \alpha \in s(x) \cap A_i,\ i = 1, 2, \ldots\},$$

and define a bijection $f\colon K'(A) \to P(A) = \prod_i S'(A_i)$ by

$$(p_i(f(x)))(\alpha) = i \cdot x(\alpha) \quad \text{for each } i \text{ and } \alpha \in A_i,$$

where p_i denotes the projection of $P(A)$ onto $S'(A_i)$. As observed in [**2**, Theorem 2], f is a homeomorphism. Then the subspace K'_∞ of $K'(A)$ defined by

$$K'_\infty = \{x \in K'(A)\colon |\{\alpha \in A\colon x(\alpha) \in Q_0\}| < \aleph_0\}$$

is contained in $H_\infty(A)$, and $f(K'_\infty) = \{y \in P(A)\colon |\{\alpha \in A\colon (p_i(y))(\alpha) \in Q_0\}| < \aleph_0\}$ which is homeomorphic to the space $K_\infty(A)$. Hence $K_\infty(A)$ is homeomorphic to $K'_\infty \subseteq H_\infty(A)$, which completes the proof of Lemma 2. Hence the proof of Theorem is also completed.

REFERENCES

1. C. H. Dowker, *An imbedding theorem for paracompact metric spaces*, Duke Math. J. **14** (1947), 639–645.
2. T. Goto, *A note on the embeddings of metric spaces into generalized Hilbert spaces*, J. Saitama Univ. **36** (1987), 1–3.
3. J. Nagata, *On the countable sum of zero-dimensional spaces*, Fund. Math. **48** (1960), 1–14.
4. ——, *A remark on general imbedding theorem in dimension theory*, Proc. Japan Acad. **39** (1963), 197–199.
5. ——, *Modern dimension theory*, Groningen, 1965.
6. E. Pol, *On the dimension of the product of metrizable spaces*, Bull. Acad. Polon. **26** (1976), 525–534.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, SAITAMA UNIVERSITY, URAWA, JAPAN