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FIXED POINT SETS OF HOMEOMORPHISMS OF METRIC PRODUCTS

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ABSTRACT. In this paper it is investigated as to when a nonempty closed subset A of a metric product X containing intervals or spheres as factors can be the fixed point set of an autohomeomorphism of X. It is shown that if X is the Hilbert cube Q or contains either the real line R or a $(2n-1)$-sphere S^{2n-1} as a factor, then A can be any nonempty closed subset. In the case where A is in $\operatorname{Int}(B^{2n+1})$, the interior of the closed unit $(2n+1)$-ball B^{2n+1}, a strong necessary condition is given. In particular, for B^3, A can neither be a nonamphicheiral knot nor a standard closed or nonplanar bordered surface.

1. Introduction. In [**9**, p. 553] a space X is defined to have *the complete invariance property* (CIP) if every nonempty closed subset of X is the fixed point set of a self-mapping of X. If this condition holds for autohomeomorphisms of X, then we shall say that X has *the complete invariance property with respect to homeomorphisms* (CIPH). In §2 it is shown that the Hilbert cube Q has CIPH as does any space Y on which S^1 acts freely provided Y possesses a bounded metric such that each orbit is (arc length metric) isometric to S^1. Some results are given in §3 which contrast the finite with the infinite dimensional case. In particular, it is shown that if a nonempty compactum A is imbedded in $\operatorname{Int}(B^{2n+1})$, then whether or not the imbedded space is the fixed point set of an autohomeomorphism of B^{2n+1} depends upon the placement of A.

A survey of results concerning CIP for metric spaces may be found in [**8**] and some nonmetric results may be found in [**3**]. Closed surfaces, B^{2n} and S^n $(n > 0)$ are known to have CIPH (see [**6**, **7**]).

The terminology used in this paper may be found in [**4** and **5**].

2. Products of arbitrary dimension.

2.1. LEMMA. *An open subset of a space having CIPH has CIPH.*

PROOF. Suppose U is an open subset of a space X which has CIPH. Let $B = X - U$ and let A be a nonempty closed subset of U. Since $X - (A \cup B) = U - A$ is open in U and hence open in X, $A \cup B$ is a closed subset of X. Thus there is an autohomeomorphism h of X whose fixed point set is $A \cup B$. It follows that $h|U$ is an autohomeomorphism of U whose fixed point set is A, and therefore U has CIPH.

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2.2. THEOREM. *If X is a metric space and A is a nonempty closed subset of $X \times S^{2n-1}$ $(X \times R)$, then there is an autohomeomorphism of $X \times S^{2n-1}$ $(X \times R)$ whose fixed point set is A (i.e., $X \times S^{2n-1}$ and $X \times R$ have CIPH).*

PROOF. We first consider the case for $X \times S^1$. Let d_1 be a bounded metric for X such that $d_1 \leq 1$ and let d_2 denote the arc length metric on S^1. Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in X \times S^1$ and let d denote the metric for $X \times S^1$ defined by

$$[d(u,v)]^2 = \sum_{i=1}^{2} [d_i(u_i, v_i)]^2.$$

Let $z = (x, p) \in X \times S^1$, $q \in S^1$ and consider the S^1-action on $X \times S^1$ defined by $zq = (x, pq)$ where pq denotes the usual multiplication of complex numbers. Given a nonempty closed subset A of $X \times S^1$, set $a(z) = \frac{1}{2}d(z, A)$. Define a mapping $f\colon X \times S^1 \to X \times S^1$ by

$$f(z) = zq, \quad \text{where } z \in X \times S^1 \quad \text{and} \quad q = \exp(ia(z)).$$

Since $0 < a(z) < 2\pi$ if $z \notin A$, it follows that $\text{Fix}(f) = A$.

To see that f is one-to-one, suppose $f(y) = f(z)$. Then y and z must lie on the same orbit and for some real number t, $y = ze^{it}$ with $d(y, z) = |t| \leq \pi$. Thus $e^{i(t+a(y))} = e^{ia(z)}$ and hence, for some integer n,

$$t + (a(y) - a(z)) = 2\pi n.$$

The triangle inequality applied to y, z, A implies

$$|t| \geq 2|a(y) - a(z)| \quad \text{and so} \quad |a(y) - a(z)| \leq \pi/2.$$

Therefore $t + (a(y) - a(z)) = 0$ and, since $|t| \geq 2|a(y) - a(z)|$, it follows that $t = 0$. Thus $y = z$ and f is one-to-one.

Since a homeomorphism of S^1 into itself must be onto, it follows that $f(\{x\} \times S^1) = \{x\} \times S^1$ for each $x \in X$, and thus f is onto. In order to conclude that f is a homeomorphism it suffices to show that f is a closed mapping. To see this, suppose B is a closed subset of $X \times S^1$ and $(u, v) \in \text{Cl}(f(B))$, the closure of $f(B)$ in $X \times S^1$. Thus there is a sequence of points $(x_1, y_1), (x_2, y_2), \ldots,$ in B such that $\lim_{n\to\infty} f(x_n, y_n) = (u, v)$. Then $\lim_{n\to\infty} x_n = u$ and, since S^1 is compact and f is one-to-one, it follows that $\lim_{n\to\infty} y_n = q$ for some $q \in S^1$. Thus $\lim_{n\to\infty}(x_n, y_n) = (u, q) \in \text{Cl}(B) = B$. Since $f(u, q) = \lim_{n\to\infty} f(x_n, y_n) = (u, v)$, it follows that $f(B) = \text{Cl}(f(B))$ and f is a closed mapping.

Now replace $X \times S^1$ by $X \times S^{2n-1}$ in the preceding proof and use the S^1-action on $X \times S^{2n-1}$ defined by

$$zq = (x, (z_1 q, \ldots, z_n q)) \quad \text{if} \quad z = (x, (z_1, \ldots, z_n)) \in X \times S^{2n-1}, \ q \in S^1.$$

If $f\colon X \times S^{2n-1} \to X \times S^{2n-1}$ is the mapping $f(z) = zq$, with $q = \exp(ia(z))$, then it follows as before that f is an autohomeomorphism with fixed point set A.

Finally, since $X \times R$ is homeomorphic to $X \times (S^1 - \{1\})$ which is an open subset of $X \times S^1$, it follows from (2.1) that $X \times R$ has CIPH.

2.3. REMARK. We remark that in the above proof, $X \times S^{2n-1}$ could be replaced by any space Y on which S^1 acts freely provided Y possesses a bounded metric such that each orbit is (arc length metric) isometric to S^1. In particular, this would apply to principle S^1-bundles.

2.4. THEOREM. *The Hilbert cube Q has CIPH.*

PROOF. We shall identify points in the plane R^2 with complex numbers and $|\ |$ shall denote the usual norm. For $n = 1, 2, \ldots$, let D_n be the disk in R^2 defined by

$$D_n = \{z \mid |z| \leq 1/2^n\}.$$

The Hilbert cube Q shall be represented by the product

$$Q = \prod_{n=1}^{\infty} D_n,$$

and we shall use the metric d on Q defined by

$$[d(x,y)]^2 = \sum_{n=1}^{\infty} |x_n - y_n|^2 \quad \text{where } x = (x_n),\ y = (y_n) \in Q.$$

Let A be a nonempty closed subset of Q. Since Q is homogeneous, without loss of generality we shall assume that $0 = (0, 0, \ldots) \in A$. For each $n = 1, 2, \ldots$, define a mapping $h_n \colon Q \to Q$ by

$$(h_n(x))_j = \begin{cases} x_j & \text{if } j \neq n, \\ x_n e^{ia(x)} & \text{if } j = n, \end{cases}$$

where $a(x)$ is as defined in the proof of (2.2). For a fixed n, and a fixed t with $0 < t \leq 1/2^n$, the subset $\{x \in Q \mid |x_n| = t\}$ is of the form $X \times S^1$. The argument used in (2.2) for that case applied simultaneously for all t (and with identity for $t = 0$) shows that h_n is an autohomeomorphism of Q. It is easy to check that the infinite left composition $h = \lim_{n\to\infty} h_n \cdots h_2 h_1$ defines an autohomeomorphism of Q. Moreover, by the construction of h, $h(x) = x$ if $x \in A$. If $x \notin A$, then $x_n \neq 0$ for some n. Thus $(h_n(x))_n \neq x_n$ and hence $h(x) \neq x$. Therefore $\mathrm{Fix}(h) = A$ as required.

A Q-manifold is a locally compact separable metric space which is locally homeomorphic to $Q \times [0, 1)$ (see [**1**, p. 18]). The following result provides some examples of Q-manifolds which have CIPH by virtue of (2.4) and (2.1).

2.5. COROLLARY. (i) *If M is a Q-manifold, then $M \times [0, 1)$ has CIPH.*

(ii) *If X is a locally compact separable ANR(metric)-space, then $X \times Q \times [0, 1)$ has CIPH.*

(iii) *A denumerable product of nondegenerate compact AR(metric)-spaces has CIPH. In particular, if X is a compact AR(metric)-space, then $X \times Q$ has CIPH.*

PROOF. (i) If M is a Q-manifold, then $M \times [0, 1)$ can be imbedded as an open subset of Q [**1**, 16.3].

(ii) If X is a locally compact separable ANR(metric)-space, then $X \times Q$ is a Q-manifold [**1**, 44.1] and therefore (ii) follows from (i).

(iii) A denumerable product of nondegenerate compact AR(metric)-spaces is homeomorphic to Q [**1**, p. 119].

3. Products of finite dimension.

3.1. PROPOSITION. *Let M be an n-manifold having a compact boundary component C with Euler characteristic $\chi(C) \neq 0$. Then M does not have CIPH.*

PROOF. Let N be a collaring of C and let $h: C \times I \to N$ be a homeomorphism such that $h(x,0) = x$ if $x \in C$. Suppose f is an autohomeomorphism of M such that $\text{Fix}(f) = M - h(C \times [0,1))$. Let $p: C \times I \to C$ denote the natural projection map and define a mapping $H: C \times I \to C$ by

$$H(x,t) = p \circ h^{-1} \circ f \circ h(x,t) \quad \text{if } x \in C,\ t \in I.$$

This yields a contradiction since H is a homotopy between a fixed point free self-mapping of C and the identity map on C.

3.2. COROLLARY. *Let M be a compact connected n-manifold without boundary having Euler characteristic $\chi(M) \neq 0$.*

(i) *If J is a subinterval of the real line R, then $M \times J$ has CIPH iff J is homeomorphic to R.*

(ii) *The cone $C(M)$ over M does not have CIPH.*

PROOF. (i) The proof is immediate by (2.2) and (3.1).

(ii) Since $M \times [0,1)$ is an open subset of $C(M)$ not having CIPH by (i), $C(M)$ does not have CIPH by (2.1).

3.3. EXAMPLE. The closed unit $(2n+1)$-cell I^{2n+1} does not have CIPH by (3.1). However, if $n > 0$, then $I^{2n+1} = U \cup V$ where U, V and $U \cap V$ are open subspaces having CIPH. Let $U = I^{2n} \times [0,1)$, $V = I^{2n} \times (0,1]$. Then $U \cap V = I^{2n} \times (0,1)$ has CIPH by (2.2). To see that U and, hence, V has CIPH, let A be a nonempty closed subset of U. Let $C = A \cup (I^{2n} \times \{1\})$. Then C is a closed subset of I^{2n+1} and $C \cap \text{Bd}(I^{2n+1}) \neq \emptyset$, where $\text{Bd}(I^{2n+1})$ denotes the manifold boundary of I^{2n+1}. It follows that there is an autohomeomorphism h of I^{2n+1} whose fixed point set is C [**6**, p. 46]. Thus $h|U$ is an autohomeomorphism of U whose fixed point set is A.

3.4. DEFINITIONS. A nonempty compact subset A in R^{2n+1} is *amphicheiral in* R^{2n+1} if there is an orientation reversing autohomeomorphism h of R^{2n+1} such that $h(A) = A$. If, in addition, $h(x) = x$ if $x \in A$, then A is *pointwise amphicheiral in* R^{2n+1}. Let $\sigma_{2n+1}: R^{2n+1} \to R^{2n+1}$ be the reflection mapping defined by

$$\sigma_{2n+1}(x_1, x_2, \ldots, x_{2n}, x_{2n+1}) = (x_1, x_2, \ldots, x_{2n}, -x_{2n+1}).$$

Since $\sigma_{2n+1}h$ is orientation preserving (reversing) iff h is an orientation reversing (preserving) autohomeomorphism of R^{2n+1}, we have the following lemma.

3.5. LEMMA. *A nonempty compact subset A in R^{2n+1} is pointwise amphicheiral in R^{2n+1} iff there is an orientation preserving autohomeomorphism h of R^{2n+1} such that $h(x) = \sigma_{2n+1}(x)$ if $x \in A$.*

We remark that a knot A is amphicheiral (in R^3) iff there is an isotopy of R^3 which deforms A onto its mirror image $\sigma_3(A)$ (see [**2**, pp. 9–10]).

3.6. THEOREM. *If a nonempty compact subset A in $\text{Int}(B^{2n+1})$ is the fixed point set of an autohomeomorphism of B^{2n+1}, then A is pointwise amphicheiral in R^{2n+1}.*

PROOF. Suppose h is an autohomeomorphism of B^{2n+1} with $\text{Fix}(h) = A$. Since $h|S^{2n}$ is fixed point free, it follows that $h|S^{2n}$ and $h|\text{Int}(B^{2n+1})$ must be orientation reversing homeomorphisms. The result follows since $\mathrm{Int}(B^{2n+1})$ is homeomorphic to R^{2n+1}.

3.7. THEOREM. *Let X be a circle, a closed orientable surface or a bordered surface other than B^2. Then there is an imbedding $h\colon X \to \mathrm{Int}(B^3)$ such that $h(X)$ is not the fixed point set of an autohomeomorphism of B^3.*

PROOF. The unit 2-sphere S^2 is not pointwise amphicheiral in R^3 since any autohomeomorphism of R^3 whose fixed point set contains S^2 must be orientation preserving. The remaining examples fail to be pointwise amphicheiral in R^3 since they contain nonempty compact subsets which are not amphicheiral in R^3. For the case of simple closed curves, if $1 < p < q$ and p, q are relatively prime integers, the torus knots $K_{p,q}$ form an infinite family of inequivalent knots which are not amphicheiral in R^3 [**5**, pp. 53–55]. Clearly, a standard (i.e., tame and unknotted) orientable closed or bordered surface with positive genus can be constructed so as to contain a copy of a torus knot, say the trefoil knot $K_{2,3}$.

For the nonorientable case, consider a standard Möbius band M obtained by attaching the ends of a rectangular strip with $2k+1$ half-twists to the boundary of the disk B^2. Assign orientations to $\mathrm{Bd}(M)$ and the unknotted center curve C of M. Since the linking numbers (see [**5**, p. 135]) $\mathrm{Lk}(\mathrm{Bd}(M), C)$ and $\mathrm{Lk}(\sigma_3(\mathrm{Bd}(M)), \sigma_3(C))$ have opposite signs and are equal to $\pm(2k+1)$, there is no orientation preserving autohomeomorphism h of R^3 such that $h(x) = \sigma_3(x)$ if $x \in M$. Thus M is not pointwise amphicheiral in R^3 by (3.5). (If $2k+1 \neq 1$, it suffices to note that $\mathrm{Bd}(M)$ is the alternating torus knot $K_{2,2k+1}$.) Since a model of every nonorientable bordered surface can be constructed from M by attaching additional rectangular strips to $\mathrm{Bd}(B^2)$ [**4**, p. 44], the nonorientable case is complete.

A planar model for a bordered surface of genus 0 with $n+1$ boundary components can be obtained by taking the disk B^2 and attaching the ends of n rectangular strips to $\mathrm{Bd}(B^2)$. If $n > 0$, then an untwisted strip can be replaced to obtain a nonplanar model which contains an annulus A with $2k$ ($k > 0$) half-twists. Since the linking numbers of the corresponding oriented boundary components of A and $\sigma_3(A)$ have opposite signs $\pm k$, A is not pointwise amphicheiral in R^3 and the proof is complete.

3.8. COMMENTS. It is shown in [**6**, p. 50] that if a nonempty compact subset A in $\mathrm{Int}(B^{2n+1})$ is contained in a hyperplane R^{2n} of R^{2n+1}, then A is the fixed point set of an autohomeomorphism of B^{2n+1}. Thus inequivalent imbeddings of a compactum into $\mathrm{Int}(B^{2n+1})$ may yield different results: A circle and a flat disk with $n > 0$ holes lying in $\mathrm{Int}(B^3)$ are fixed point sets of autohomeomorphisms of B^3, but the proof of (3.7) shows that each has infinitely many inequivalent homeomorphs which are not. It should also be noted that nonplanar simple closed curves such as the figure-eight knot, which is both amphicheiral and invertible (see [**2**, pp. 10–11]), must be pointwise amphicheiral in R^3.

3.9. QUESTION. *Is the necessary condition in* (3.6) *also sufficient?*

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