INTEGRAL BROWN-GITLER SPECTRA
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(Communicated by Haynes R. Miller)

ABSTRACT. A Thom spectrum model for integral Brown-Gitler spectra is established and shown to have a multiplicative property. This clarifies certain aspects of an earlier application to splitting $b_0 \wedge b_0$.

1. Statement of results. Brown-Gitler spectra have had many important applications in homotopy theory, most notably in [M1 and C1]. They were originally constructed in [BG] by a complicated Postnikov argument, but a Thom spectrum model suggested in [M1] and established to be correct in [C2] made them more down-to-earth.

Integral Brown-Gitler spectra at the prime 2 were introduced in [M2], where they were useful in a splitting of $b_0 \wedge b_0$. A Thom spectrum model was suggested there, and an expanded account, including both Thom spectrum and Postnikov models, was presented in [Sh]. The odd-primary version of the Thom space model was discussed in [Ka]. In none of these is the base space for these Thom spectra explicitly defined. The purpose of this paper is to clarify the Thom spectrum model of integral Brown-Gitler spectra.

Recall that there is an isomorphism of Hopf algebras

\[ H_*(\Omega^2 S^3) \cong E[x_j : j \geq 0] \otimes F_p[y_j : j \geq 1] \] if $p$ odd

with $|x_j| = 2p^j - 1$ and $|y_j| = 2p^j - 2$. The only modification required for $p = 2$ is $x_j^2 = y_{j+1}$. All homology groups have coefficients in the field $F_p$ with $p$ elements, unless indicated otherwise. Define a weight on the monomials in $H_*(\Omega^2 S^3)$ by

\[ \text{wt}(x_j) = \text{wt}(y_j) = p^j, \quad \text{wt}(ab) = \text{wt}(a) + \text{wt}(b). \]

The space $\Omega^2 S^3$ admits an increasing filtration by spaces $F_n \Omega^2 S^3$, due to May and Milgram [May, Mil], such that $H_*(F_n \Omega^2 S^3) \subset H_*(\Omega^2 S^3)$ is the span of monomials of weight $\leq n$ [CLM, p. 239].

Let $S^3(3)$ denote the 3-connected cover of $S^3$. Then there is a homotopy fibration

\[ \Omega^2 S^3(3) \rightarrow \Omega^2 S^3 \rightarrow S^1. \]

$\Omega^2 S^3(3)$ was called $W$ in [DGM and M2]. Using the multiplication on $\Omega^2 S^3$, one easily deduces $\Omega^2 S^3 \cong S^1 \times \Omega^2 S^3(3)$, and so $H_*(\Omega^2 S^3(3)) \subset H_*(\Omega^2 S^3)$ is the span
of monomials of weight divisible by $p$. The filtration on $\Omega^2 S^3$ induces a filtration on $H_*(\Omega^2 S^3(3))$ by

$$F_n H_*(\Omega^2 S^3(3)) = H_*(F_n \Omega^2 S^3) \cap H_*(\Omega^2 S^3(3)),$$

the span of monomials of weight $\leq n$ and divisible by $p$. In [M2] for $p = 2$ and [Ka] for odd $p$, it was asserted without proof that $F_n H_*(\Omega^2 S^3(3))$ is induced by an actual filtration of the space $\Omega^2 S^3(3)$. This does not follow for general reasons, but we shall show that it can be achieved after localization with respect to mod $p$ homology.

We denote by $X_p$ the Bousfield localization [Bo] of the space $X$ with respect to the homology theory $H_*(-;F_p) = H_{F_p}$. For a fixed prime $p$, let $\mathcal{F}_n = (F_n \Omega^2 S^3)_p$. There are product maps

$$\mathcal{F}_m \times \mathcal{F}_n \to \mathcal{F}_{m+n}$$

induced by the corresponding maps for the filtration spaces and the fact that localization preserves finite products [Bo, 12.5]. Define $A_n$ by the homotopy fibration sequence

$$A_n \to \mathcal{F}_{pn+1} \to S^1,$$  

where the second map is the localization of the composite

$$F_{pn+1} \Omega^2 S^3 \to \Omega^2 S^3 \to S^1.$$ 

It follows easily from the definitions and [Bo, 12.7] that the space $A_n$ is $H_{F_p}$-local.

Most of our effort is directed toward the following result, which is proved in §2.

**Theorem 1.3.** The fiber sequence (1.2) is equivalent to a product fibration. Indeed, there is a map $A_n \to \mathcal{F}_{pn}$ and a commutative diagram of fibrations

$$\begin{array}{ccc}
A_n & \to & A_n \\
\downarrow & & \downarrow \\
S^1_p \times A_n & \to & S^1_p \times \mathcal{F}_{pn} = \mathcal{F}_1 \times \mathcal{F}_{pn} \to \mathcal{F}_{pn+1} \\
\downarrow_{p^1} & & \downarrow \\
S^1_p & \to & S^1_p
\end{array}$$

which is an equivalence on total spaces and on fibers.

Then $H_*(A_n) \approx F_{pn} H_*(\Omega^2 S^3(3))$, the span of monomials of weight $p^i$ with $i \leq n$.

**Remark.** That $H_*(A_n)$ is as claimed is not immediate from (1.2) and the Serre spectral sequence, since it is not clear a priori that the fibration of (1.2) is orientable. For example, the fibration $\mathcal{F}_{p^k} \to S^1_p$ is not orientable. Our Theorem 1.3 establishes the orientability in (1.2) indirectly.

In [M3], Mahowald constructed a stable spherical fibration $\xi$ over $\Omega^2 S^3$ such that

(i) the Thom spectrum $T(\xi)$ is equivalent to the mod $p$ Eilenberg-Mac Lane spectrum $H\mathbb{Z}/p$,

(ii) the Thom spectrum $T(\xi | \Omega^2 S^3(3)_p)$ is equivalent to the $p$-complete Eilenberg-Mac Lane spectrum $H\mathbb{Z}_p$, and

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the Thom spectrum \( T(\xi \mid F_n \Omega^2 S^3_p) \) is equivalent to the \( n \)th mod \( p \) Brown-Gitler spectrum \( B(n) \).

Expanded accounts appear in [Ka and CMT].

We define \( \tilde{B}_1(n) \), the \( n \)th integral Brown-Gitler spectrum at \( p \), to be the Thom spectrum \( T(\xi \mid A_n) \), using the commutative diagram

\[
\begin{array}{c}
A_n \\
\downarrow \iota_n \\
\Omega^2 S^3(3)_p \\
\end{array} \longrightarrow \begin{array}{c}
F_{pn+1} \\
\end{array}
\]

(1.4)

Thomifying the composites

\[
A_m \times A_n \xrightarrow{\phi_m \times \phi_n} F_{pm} \times F_{pn} \rightarrow F_{pm+pn} \rightarrow F_{pm+pn+1} \rightarrow A_{m+n},
\]

where the last map splits the equivalence of 1.3, yields pairings \( B_1(m) \wedge B_1(n) \rightarrow B_1(m+n) \). These pairings played a crucial role in the application to splitting \( bo \wedge bo \) in [DGM, Ka and M2]. The clarification of their existence is a major reason for the care in this work.

It is clear from 1.3 that \( T(\iota_n): B_1(n) \rightarrow H\mathbb{Z}_p \) induces a monomorphism in homology. Recall Milnor's description

\[
H_*(H\mathbb{Z}_p) = E[\chi(\tau_j): j \geq 1] \otimes F_p[\chi(\xi_j): j \geq 1],
\]

where \( |\tau_j| = 2p^j - 1 \), \( |\xi_j| = 2p^j - 2 \), and \( \chi \) denotes the canonical antiautomorphism of the dual of the mod \( p \) Steenrod algebra \( A \). The usual modification \( \tau_j^2 = \xi_{j+1} \) applies when \( p = 2 \). We define a weight by

\[
\text{wt}(\chi(\tau_j)) = \text{wt}(\chi(\xi_j)) = p^j, \quad \text{wt}(ab) = \text{wt}(a) + \text{wt}(b).
\]

Note that all monomials have weight divisible by \( p \). The relationship between these classes and those in (1.1) under the Thom isomorphism was discussed in [CMT].

The first two parts of the following theorem, which states the basic properties of integral Brown-Gitler spectra, now follow immediately from Theorem 1.3.

**THEOREM 1.5.** For \( n > 0 \), there is a \( p \)-complete spectrum \( B_1(n) \) and a map

\[
B_1(n) \xrightarrow{g_*} H\mathbb{Z}_p
\]

such that

(i) \( g_* \) sends \( H_*(B_1(n)) \) isomorphically onto the span of monomials of weight \( \leq pn \);

(ii) there are pairings

\[
B_1(m) \wedge B_1(n) \rightarrow B_1(m+n)
\]

whose homology homomorphism is compatible with the multiplication in \( H_*(H\mathbb{Z}_p) \);

(iii) for any CW complex \( X \),

\[
g_*: B_1(n)_i(X) \rightarrow H_i(X; \mathbb{Z}_p)
\]

is surjective if \( i \leq 2p(n+1) - 1 \).
Part (iii) is not easily proved from our perspective, but does follow from a straightforward adaptation of the proof of [Sh, 5.1], which was given only for $p = 2$. This adaptation requires a map from $B_1(n)$ into the mod $p$ Brown-Gitler spectrum $B(pn + 1)$ inducing the obvious inclusion in homology. Such a map follows immediately from the definitions and (1.4). The argument of [Sh] then allows us to deduce the surjectivity of $B_1(n)_*(X) \to H_*(X; \mathbb{Z}_p)$ from that of $B(pn + 1)_*(X) \to H_*(X; \mathbb{F}_p)$.

Many readers may be more familiar with the cohomology criterion

$$H^*(B_1(n)) \approx \mathcal{A}/\mathcal{A}(\beta, \chi^{P^i}: i > n).$$

This is easily seen to be dual to (i) above. We also remark that our indexing differs from that of [Sh and Ka], who would call our spectrum $B_1(pn + 1)$. We thank Don Shimamoto for helpful comments.

2. Proof of Theorem 1.3. We begin by showing that the inclusion $F_{m-1} \Omega^2 S^3 \to F_m \Omega^2 S^3$ may be considered as the inclusion into a mapping cone. Let $\Sigma_m$ denote the symmetric group on $m$ letters, and $F(\mathbb{R}^2, m)$ the space of $m$-tuples of distinct points in $\mathbb{R}^2$. If $X$ is a pointed $\Sigma_m$-space, we define

$$M_m(X) = F(\mathbb{R}^2, m) \times_{\Sigma_m} X / F(\mathbb{R}^2, m) \times_{\Sigma_m} *.$$

Let $I$ denote the unit interval, $\hat{I}$ its boundary, and $\partial I^m$ the boundary of $I^m$.

**Lemma 2.1.** Let $F_m = F_m \Omega^2 S^3$. There is a cofibration sequence

$$M_m(\partial I^m) \to F_{m-1} \to F_m.$$

**Remark.** Extending this cofibration shows that $\Sigma M_m(\partial I^m) \simeq F_m/F_{m-1}$. In particular, $H_*(M_m(\partial I^m))$ is clear from the cofibration.

**Proof.** Let $T^m(I/\hat{I})$ denote the fat wedge, consisting of points in the $m$-fold Cartesian product having at least one component the basepoint. Viewing $I^m$ as the cone $C(\partial I^m)$ yields a $\Sigma_m$-equivariant cofibration

$$\partial I^m \to T^m(I/\hat{I}) \hookrightarrow (I/\hat{I}) \times^m,$$

and hence a cofibration

$$(2.2) M_m(\partial I^m) \xrightarrow{k} M_m(T^m(I/\hat{I})) \to M_m((I/\hat{I}) \times^m).$$

Recall from [May] that

$$F_m = \left( \bigcup_{k \leq m} M_k((I/\hat{I}) \times^k) \right) / \sim,$$

where, letting * denote omission,

$$(2.3) (x_1, \ldots, x_k, t_1, \ldots, t_k) \sim (x_1, \ldots, \hat{x}_i, \ldots, x_k, t_i, \ldots, \hat{t}_i, \ldots, t_k)$$

if $t_i = \ast \in I/\hat{I}$.

The map $c$ of Lemma 2.1 is the composite

$$M_m(\partial I^m) \to M_m(T^m(I/\hat{I})) \to F_{m-1},$$

where the second map uses the equivalence relation (2.3) to ignore the basepoint in at least one component. The required homeomorphism from the mapping cone...
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of $c$ to $F_m$ is a quotient of the homeomorphism in (2.2) from the mapping cone of $k$ to $M_m((I/I)^{\times m})$. \hfill \Box

Now we return to the proof of Theorem 1.3. Assume by induction that the theorem has been proved for $n - 1$. Note that $\mathcal{F}_{p(n-1)+1} \to \mathcal{F}_{p(n-1)}$ is an equivalence. Localizing Lemma 2.1 yields a map

$$M_{pn}(\partial I^{pn})_p \to \mathcal{F}_{p(n-1)} \simeq S^1_p \times A_{n-1},$$

and, since $H^1(M_{pn}(\partial I^{pn})_p; \mathbb{Z}_p) = 0$, the map is of the form $* \times h$. Let $Y$ denote the mapping cone of $h$. The map of cofibrations

$$M_{pn}(\partial I^{pn})_p \to \mathcal{F}_{p(n-1)} \to \mathcal{F}_{pn} \to \mathcal{F}_{pn+1}$$

shows that $H_*Y \to H_*\mathcal{F}_{pn}$ is injective with image spanned by monomials of weight divisible by $p$. There is a map of fibrations

$$Y \to A_n$$

$$S^1_p \times Y \to S^1_p \times \mathcal{F}_{pn} = S^1_p \times \mathcal{F}_{pn+1}$$

The map of total spaces induces an isomorphism in mod $p$ homology, and, since $\mathcal{F}_{pn+1}$ is $HF_{p*}$-local, this map is an $HF_{p*}$-localization, and so there is an equivalence of fibrations

$$Y_p \to A_n$$

$$S^1_p \times Y_p \to \mathcal{F}_{pn+1}$$

extending the induction, and completing the proof of Theorem 1.3. \hfill \Box

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