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INTEGRAL BROWN-GITLER SPECTRA

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ABSTRACT. A Thom spectrum model for integral Brown-Gitler spectra is established and shown to have a multiplicative property. This clarifies certain aspects of an earlier application to splitting $bo \wedge bo$.

1. Statement of results. Brown-Gitler spectra have had many important applications in homotopy theory, most notably in [**M1** and **C1**]. They were originally constructed in [**BG**] by a complicated Postnikov argument, but a Thom spectrum model suggested in [**M1**] and established to be correct in [**C2**] made them more down-to-earth.

Integral Brown-Gitler spectra at the prime 2 were introduced in [**M2**], where they were useful in a splitting of $bo \wedge bo$. A Thom spectrum model was suggested there, and an expanded account, including both Thom spectrum and Postnikov models, was presented in [**Sh**]. The odd-primary version of the Thom space model was discussed in [**Ka**]. In none of these is the base space for these Thom spectra explicitly defined. The purpose of this paper is to clarify the Thom spectrum model of integral Brown-Gitler spectra.

Recall that there is an isomorphism of Hopf algebras

$$H_*(\Omega^2 S^3) \approx E[x_j\colon\ j \geq 0] \otimes \mathbf{F}_p[y_j\colon\ j \geq 1] \quad \text{if } p \text{ odd} \tag{1.1}$$

with $|x_j| = 2p^j - 1$ and $|y_j| = 2p^j - 2$. The only modification required for $p = 2$ is $x_j^2 = y_{j+1}$. All homology groups have coefficients in the field $\mathbf{F}_p$ with p elements, unless indicated otherwise. Define a weight on the monomials in $H_*(\Omega^2 S^3)$ by

$$\mathrm{wt}(x_j) = \mathrm{wt}(y_j) = p^j, \qquad \mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b).$$

The space $\Omega^2 S^3$ admits an increasing filtration by spaces $F_n\Omega^2 S^3$, due to May and Milgram [**May**, **Mil**], such that $H_*(F_n\Omega^2 S^3) \subset H_*(\Omega^2 S^3)$ is the span of monomials of weight $\leq n$ [**CLM**, p. 239].

Let $S^3\langle 3\rangle$ denote the 3-connected cover of S^3. Then there is a homotopy fibration

$$\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to S^1.$$

$\Omega^2 S^3\langle 3\rangle$ was called W in [**DGM** and **M2**]. Using the multiplication on $\Omega^2 S^3$, one easily deduces $\Omega^2 S^3 \simeq S^1 \times \Omega^2 S^3\langle 3\rangle$, and so $H_*(\Omega^2 S^3\langle 3\rangle) \subset H_*(\Omega^2 S^3)$ is the span

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of monomials of weight divisible by p. The filtration on $\Omega^2 S^3$ induces a filtration on $H_*(\Omega^2 S^3\langle 3\rangle)$ by

$$F_n H_*\Omega^2 S^3\langle 3\rangle = H_*(F_n\Omega^2 S^3) \cap H_*(\Omega^2 S^3\langle 3\rangle),$$

the span of monomials of weight $\le n$ and divisible by p. In [**M2**] for $p = 2$ and [**Ka**] for odd p, it was asserted without proof that $F_n H_*\Omega^2 S^3\langle 3\rangle$ is induced by an actual filtration of the space $\Omega^2 S^3\langle 3\rangle$. This does not follow for general reasons, but we shall show that it can be achieved after localization with respect to mod p homology.

We denote by X_p the Bousfield localization [**Bo**] of the space X with respect to the homology theory $H_*(-;\mathbf{F}_p) = H\mathbf{F}_{p*}$. For a fixed prime p, let $\mathcal{F}_n = (F_n\Omega^2 S^3)_p$. There are product maps

$$\mathcal{F}_m \times \mathcal{F}_n \xrightarrow{\mu} \mathcal{F}_{m+n}$$

induced by the corresponding maps for the filtration spaces and the fact that localization preserves finite products [**Bo**, 12.5]. Define A_n by the homotopy fibration sequence

$$A_n \to \mathcal{F}_{pn+1} \to S^1_p, \tag{1.2}$$

where the second map is the localization of the composite

$$F_{pn+1}\Omega^2 S^3 \to \Omega^2 S^3 \to S^1.$$

It follows easily from the definitions and [**Bo**, 12.7] that the space A_n is $H\mathbf{F}_{p*}$-local.

Most of our effort is directed toward the following result, which is proved in §2.

THEOREM 1.3. *The fiber sequence* (1.2) *is equivalent to a product fibration. Indeed, there is a map $A_n \xrightarrow{\phi} \mathcal{F}_{pn}$ and a commutative diagram of fibrations*

$$\begin{tikzcd}
A_n \arrow[rr] \arrow[d] & & A_n \arrow[d] \\
S^1_p \times A_n \arrow[r, "1\times\phi"] \arrow[d, "p_1"] & S^1_p \times \mathcal{F}_{pn} = \mathcal{F}_1 \times \mathcal{F}_{pn} \arrow[r, "\mu"] & \mathcal{F}_{pn+1} \arrow[d] \\
S^1_p \arrow[rr, equal] & & S^1_p
\end{tikzcd}$$

which is an equivalence on total spaces and on fibers.

Then $H_*(A_n) \approx F_{pn}H_*(\Omega^2 S^3\langle 3\rangle)$, the span of monomials of weight pi with $i \le n$.

REMARK. That H_*A_n is as claimed is not immediate from (1.2) and the Serre spectral sequence, since it is not clear *a priori* that the fibration of (1.2) is orientable. For example, the fibration $\mathcal{F}_{p^k} \to S^1_p$ is not orientable. Our Theorem 1.3 establishes the orientability in (1.2) indirectly.

In [**M3**], Mahowald constructed a stable spherical fibration ξ over $\Omega^2 S^3_p$ such that

(i) the Thom spectrum $T(\xi)$ is equivalent to the mod p Eilenberg-Mac Lane spectrum $H\mathbf{Z}/p$,

(ii) the Thom spectrum $T(\xi \mid \Omega^3 S^3\langle 3\rangle_p)$ is equivalent to the p-complete Eilenberg-Mac Lane spectrum $H\mathbf{Z}_p^\wedge$, and

(iii) the Thom spectrum $T(\xi \mid F_n\Omega^2 S^3_p)$ is equivalent to the nth mod p Brown-Gitler spectrum $B(n)$.

Expanded accounts appear in [**Ka** and **CMT**].

We define $B_1(n)$, the nth integral Brown-Gitler spectrum at p, to be the Thom spectrum $T(\xi \mid A_n)$, using the commutative diagram

$$\begin{tikzcd} A_n \arrow[r] \arrow[d, "i_n"'] & \mathcal{F}_{pn+1} \arrow[d] \\ \Omega^2 S^3\langle 3\rangle_p \arrow[r] & \Omega^2 S^3_p. \end{tikzcd} \tag{1.4}$$

Thomifying the composites

$$A_m \times A_n \xrightarrow{\phi_m \times \phi_n} \mathcal{F}_{pm} \times \mathcal{F}_{pn} \to \mathcal{F}_{pm+pn} \to \mathcal{F}_{pm+pn+1} \to A_{m+n},$$

where the last map splits the equivalence of 1.3, yields pairings $B_1(m) \wedge B_1(n) \to B_1(m+n)$. These pairings played a crucial role in the application to splitting $bo \wedge bo$ in [**DGM**, **Ka** and **M2**]. The clarification of their existence is a major reason for the care in this work.

It is clear from 1.3 that $T(i_n)\colon B_1(n) \to H\mathbf{Z}_p^\wedge$ induces a monomorphism in homology. Recall Milnor's description

$$H_*(H\mathbf{Z}_p^\wedge) = E[\chi(\tau_j)\colon\ j \geq 1] \otimes \mathbf{F}_p[\chi(\xi_j)\colon\ j \geq 1],$$

where $|\tau_j| = 2p^j - 1$, $|\xi_j| = 2p^j - 2$, and χ denotes the canonical antiautomorphism of the dual of the mod p Steenrod algebra $\mathcal{A}$. The usual modification $\tau_j^2 = \xi_{j+1}$ applies when $p = 2$. We define a weight by

$$\mathrm{wt}(\chi(\tau_j)) = \mathrm{wt}(\chi(\xi_j)) = p^j, \qquad \mathrm{wt}(ab) = \mathrm{wt}(a) + \mathrm{wt}(b).$$

Note that all monomials have weight divisible by p. The relationship between these classes and those in (1.1) under the Thom isomorphism was discussed in [**CMT**].

The first two parts of the following theorem, which states the basic properties of integral Brown-Gitler spectra, now follow immediately from Theorem 1.3.

THEOREM 1.5. *For $n > 0$, there is a p-complete spectrum $B_1(n)$ and a map*

$$B_1(n) \xrightarrow{g} H\mathbf{Z}_p^\wedge$$

such that

(i) *g_* sends $H_*(B_1(n))$ isomorphically onto the span of monomials of weight $\leq pn$;*

(ii) *there are pairings*

$$B_1(m) \wedge B_1(n) \to B_1(m+n)$$

whose homology homomorphism is compatible with the multiplication in $H_*(H\mathbf{Z}_p^\wedge)$;

(iii) *for any CW complex X,*

$$g_*\colon\ B_1(n)_i(X) \to H_i(X; \mathbf{Z}_p^\wedge)$$

is surjective if $i \leq 2p(n+1) - 1$.

Part (iii) is not easily proved from our perspective, but does follow from a straightforward adaptation of the proof of [**Sh**, 5.1], which was given only for $p = 2$. This adaptation requires a map from $B_1(n)$ into the mod p Brown-Gitler spectrum $B(pn+1)$ inducing the obvious inclusion in homology. Such a map follows immediately from the definitions and (1.4). The argument of [**Sh**] then allows us to deduce the surjectivity of $B_1(n)_*(X) \to H_*(X; \hat{\mathbf{Z}_p})$ from that of $B(pn+1)_*(X) \to H_*(X; \mathbf{F}_p)$.

Many readers may be more familiar with the cohomology criterion

$$H^*(B_1(n)) \approx \mathcal{A}/\mathcal{A}(\beta, \chi P^i\colon\ i > n).$$

This is easily seen to be dual to (i) above. We also remark that our indexing differs from that of [**Sh** and **Ka**], who would call our spectrum $B_1(pn+1)$. We thank Don Shimamoto for helpful comments.

2. Proof of Theorem 1.3. We begin by showing that the inclusion $F_{m-1}\Omega^2 S^3 \to F_m \Omega^2 S^3$ may be considered as the inclusion into a mapping cone. Let Σ_m denote the symmetric group on m letters, and $F(\mathbf{R}^2, m)$ the space of m-tuples of distinct points in $\mathbf{R}^2$. If X is a pointed Σ_m-space, we define

$$M_m(X) = F(\mathbf{R}^2, m) \times_{\Sigma_m} X / F(\mathbf{R}^2, m) \times_{\Sigma_m} *.$$

Let I denote the unit interval, $\dot{I}$ its boundary, and ∂I^m the boundary of I^m.

LEMMA 2.1. *Let $F_m = F_m\Omega^2 S^3$. There is a cofibration sequence*

$$M_m(\partial I^m) \xrightarrow{c} F_{m-1} \to F_m.$$

REMARK. Extending this cofibration shows that $\Sigma M_m(\partial I^m) \simeq F_m/F_{m-1}$. In particular, $H_*(M_m(\partial I^m))$ is clear from the cofibration.

PROOF. Let $T^m(I/\dot{I})$ denote the fat wedge, consisting of points in the m-fold Cartesian product having at least one component the basepoint. Viewing I^m as the cone $C(\partial I^m)$ yields a Σ_m-equivariant cofibration

$$\partial I^m \to T^m(I/\dot{I}) \hookrightarrow (I/\dot{I})^{\times m},$$

and hence a cofibration

$$M_m(\partial I^m) \xrightarrow{k} M_m(T^m(I/\dot{I})) \to M_m((I/\dot{I})^{\times m}). \tag{2.2}$$

Recall from [**May**] that

$$F_m = \Big(\bigcup_{k \le m} M_k((I/\dot{I})^{\times k})\Big)/\sim,$$

where, letting ˆ denote omission,

$$(x_1, \ldots, x_k, t_1, \ldots, t_k) \sim (x_1, \ldots, \hat{x}_i, \ldots, x_k, t_i, \ldots, \hat{t}_i, \ldots, t_k) \text{ if } t_i = * \in I/\dot{I}. \tag{2.3}$$

The map c of Lemma 2.1 is the composite

$$M_m(\partial I^m) \to M_m(T^m(I/\dot{I})) \to F_{m-1},$$

where the second map uses the equivalence relation (2.3) to ignore the basepoint in at least one component. The required homeomorphism from the mapping cone of c to F_m is a quotient of the homeomorphism in (2.2) from the mapping cone of k to $M_m((I/\dot{I})^{\times m})$. $\square$

Now we return to the proof of Theorem 1.3. Assume by induction that the theorem has been proved for $n-1$. Note that $\mathcal{F}_{p(n-1)+1} \to \mathcal{F}_{pn-1}$ is an equivalence. Localizing Lemma 2.1 yields a map

$$M_{pn}(\partial I^{pn})_p \to \mathcal{F}_{pn-1} \simeq S^1_p \times A_{n-1},$$

and, since $H^1(M_{pn}(\partial I^{pn})_p; \mathbf{Z}_p^\frown) = 0$, the map is of the form $* \times h$. Let Y denote the mapping cone of h. The map of cofibrations

$$\begin{tikzcd} M_{pn}(\partial I^{pn})_p \arrow[r, "h"] \arrow[d, equal] & A_{n-1} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "\phi"'] & \mathcal{F}_{pn}/\mathcal{F}_{pn-1} \arrow[d, equal] \\ M_{pn}(\partial I^{pn})_p \arrow[r] & \mathcal{F}_{pn-1} \arrow[r] & \mathcal{F}_{pn} \arrow[r] & \mathcal{F}_{pn}/\mathcal{F}_{pn-1} \end{tikzcd}$$

shows that $H_*Y \to H_*\mathcal{F}_{pn}$ is injective with image spanned by monomials of weight divisible by p. There is a map of fibrations

$$\begin{tikzcd} Y \arrow[rr] \arrow[d] & & A_n \arrow[d] \\ S^1_p \times Y \arrow[r, "1\times\phi"] \arrow[d, "p_1"] & S^1_p \times \mathcal{F}_{pn} = \mathcal{F}_1 \times \mathcal{F}_{pn} \arrow[r, "\mu"] & \mathcal{F}_{pn+1} \arrow[d] \\ S^1_p \arrow[rr, equal] & & S^1_p. \end{tikzcd}$$

The map of total spaces induces an isomorphism in mod p homology, and, since $\mathcal{F}_{pn+1}$ is $H\mathbf{F}_{p*}$-local, this map is an $H\mathbf{F}_{p*}$-localization, and so there is an equivalence of fibrations

$$\begin{tikzcd} Y_p \arrow[r] \arrow[d] & A_n \arrow[d] \\ S^1_p \times Y_p \arrow[r] \arrow[d] & \mathcal{F}_{pn+1} \arrow[d] \\ S^1_p \arrow[r, equal] & S^1_p, \end{tikzcd}$$

extending the induction, and completing the proof of Theorem 1.3. $\square$

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