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NEIGHBORHOODS OF POINTS IN CODIMENSION-ONE SUBMANIFOLDS LIE IN CODIMENSION-ONE SPHERES

FREDRIC D. ANCEL

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ABSTRACT. For $n \geq 4$, let M be an $(n-1)$-manifold embedded in an n-manifold N. For each point p of M, there is an $(n-1)$-sphere Σ in N such that $\Sigma \cap M$ is a neighborhood of p in M.

We work in the category of topological manifolds without boundary and topological embeddings.

The $n = 3$ case of this result is established by Theorem 5 of [**2**]. A weak version of this result for $n \geq 5$ is found in Theorem 5B.10 of [**3**]. This particular proof arose in response to a private query from D. L. Loveland of Utah State University.

This type of result is used to generalize theorems concerning local properties of wild codimension-one spheres into theorems about arbitrary wild codimension-one submanifolds. One application is found in Theorem 6 of [**2**]. Another application is discussed on pages 38 and 39 of [**1**].

PROOF. Without loss of generality, we can cut M and N down to assure that both are orientable. This makes any embedding of M in N 2-sided. Now, for an open subset V of M, an embedding $e: V \to N$ is *tame* if there is an embedding $E: V \times \mathbf{R} \to N$ such that $E(x, 0) = e(x)$ for each $x \in V$; the embedding E is called a *collar* of e.

Let $\{U_i : i \geq 0\}$ be a decreasing sequence of open neighborhoods of p in N with diameters converging to zero. Let $\{D_i : i \geq 0\}$ be a sequence of $(n-1)$-balls in M such that for each $i \geq 0$, $\{p\} \cup D_{i+1} \subset \text{int}(D_i)$ and $D_i \subset U_i$. For $0 \leq i < j < \infty$, let $A(i,j) = (\text{int}(D_i)) - D_j$ and let $A(i, \infty) = (\text{int}(D_i)) - \{p\}$.

Let $e_0: M \to N$ denote the given inclusion. Repeated applications of Theorem 2.2 of [**1**] yields a sequence of embeddings $e_i: M \to N$ which, for each $i \geq 1$, satisfy the following three conditions.

(1) $e_i = e_{i-1}$ on $M - A(i-1, i+1)$.

(2) $e_i | A(0, i+1)$ is tame.

(3) $e_i(D_j) \subset U_j$ for each $j \geq 0$.

It follows that the sequence $\{e_i\}$ converges to an embedding $f: M \to N$ such that for each $i \geq 0$, $f = e_i$ on $M - A(i, \infty)$. Consequently, $f | A(0, \infty)$ is tame.

According to [**4**], f cannot have isolated wild points. Hence, $f|\text{int}(D_0)$ is, in fact, tame. Thus, $f|\text{int}(D_0)$ has a collar, which we use to slide f to an embedding $g: M \to N$ such that $g(\text{int}(D_0)) \cap f(\text{int}(D_0)) = \varnothing$ and $g = f$ on $M - \text{int}(D_0)$.

As $p = f(p) \in f(\text{int}(D_0))$, there is an $i \geq 0$ such that $g(M) \cap U_i = \varnothing$. Since $e_i(D_i) \subset U_i$ and $e_i = f$ on $M - \text{int}(D_i)$, then $e_i(\text{int}(D_0)) \cap g(\text{int}(D_0)) = \varnothing$ and

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$e_i = g$ on ∂D_0. So $\Sigma = e_i(D_0) \cup g(D_0)$ is an $(n-1)$-sphere. Since $e_i = e_0$ on D_{i+1}, then $D_{i+1} \subset \Sigma \cap M$. □

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF WISCONSIN-MILWAUKEE, WISCONSIN 53201