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NON-G-EQUIVALENT MOORE G-SPACES OF THE SAME TYPE

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ABSTRACT. Let G be a finite group. By a Moore G-space we mean a G-space X such that for each subgroup H of G the fixed point space X^H is a Moore space of type (M_H, n), where $n > 1$ is a fixed integer and M_H are abelian groups. In this paper it is shown that there exist infinitely many non-G-homotopy equivalent Moore G-spaces of certain given type.

1. Introduction. Let G be a finite group. G-spaces, G-actions, G-maps and G-homotopies considered in this paper will be pointed. We shall work in the category of G-spaces having the G-homotopy type of a G-CW-complex [**B**], and we make tacit use of the standard strategies for keeping our constructions within this category.

Let O_G be the category of canonical orbits of G, whose objects are the left coset spaces G/H, H a subgroup of G, and whose morphisms are the equivariant (with respect to left translation) maps $G/H \to G/K$. A *coefficient system for* G is a contravariant functor from O_G into the category of abelian groups. A coefficient system will be called *rational* if its range is the category of $\mathbf{Q}$-vector spaces. For a G-space X, coefficient systems $\underline{\pi}_n(X)$ and $\underline{\tilde{H}}_n(X)$ can be defined by $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$, $\underline{\tilde{H}}_n(X)(G/H) = \tilde{H}_n(X^H)$, where $\tilde{H}_n(\ )$ denotes the reduced singular homology with $\mathbf{Z}$-coefficients.

Let M be a coefficient system for G and $n \geq 2$ an integer.

DEFINITION. A *Moore G-space* of type (M, n) is a G-space X such that each fixed point space X^H is 1-connected and

$$\underline{\tilde{H}}_q(X) = \begin{cases} M & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Coefficient systems for G and their natural transformations form an abelian category with sufficiently many projectives and injectives [**B**]. The same holds for rational coefficient systems. Let M be a rational coefficient system and $n \geq 2$ an integer. By a result of P. J. Kahn [**K**], if $\operatorname{proj.dim} M < n$, then, up to G-equivalence (=G-homotopy equivalence), there exists exactly one Moore G-space of type (M, n).

In contrast to the non-equivariant case, uniqueness does not hold in general for Moore G-spaces. In [**K**] there is given an example of a rational coefficient system M and two Moore G-spaces L_1, L_2 of type $(M, 2)$ that are not G-equivalent.

In this paper we extend the above result showing, by quite a different method, that there exist infinitely many non-G-equivalent Moore G-spaces of type $(M, 2)$.

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2. G-spaces with two nontrivial systems of homotopy groups. Suppose that X is a G-simple G-space (i.e. each X^H is connected and simple) having only two nontrivial systems of homotopy groups $\underline{\pi}_n(X)$ and $\underline{\pi}_m(X)$ with $n < m$. In such a situation X is determined by its equivariant k-invariant $k(X)$ which lies in the Bredon cohomology group $\tilde{H}_G^{m+1}(K(\underline{\pi}_n(X), n), \underline{\pi}_m(X))$, where $K(\underline{\pi}_n(X), n)$ is an Eilenberg-MacLane G-space of type $(\underline{\pi}_n(X), n)$ [**T**]. Let us denote $\underline{\pi}_n(X) = N$, $\underline{\pi}_m(X) = M$. The sets $R = [K(N,n), K(N,n)]_G$ and

$$L = [K(M, m+1), K(M, m+1)]_G$$

of G-homotopy classes of G-maps are rings with 1, the multiplication being induced by composition of maps. Let $U(L)$, $U(R)$ denote the groups of units (i.e. of classes of G-equivalences) of L and R, respectively. Define an action of the group $U(L) \times U(R)$ on $[K(N,n), K(M, m+1)]_G$ by the formula $(\alpha, \beta)u = \alpha \circ u \circ \beta^{-1}$, where $\circ$ is induced by composition of maps. Let $V(N, n, M, m+1)$ denote the orbit space of $[K(N,n), K(M, m+1)]_G$ under the above $U(L) \times U(R)$-action.

PROPOSITION 1. *The set $V(N, n, M, m+1)$ is in one-to-one correspondence with the set of G-homotopy types of G-simple G-spaces X having only two nontrivial systems of homotopy groups $\underline{\pi}_n(X) = N$, $\underline{\pi}_m(X) = M$.*

PROOF. From G-homotopy invariance of G-pullbacks, which we get applying formally a dual of Theorem 6.6 of [**H**] to the G-equivariant case, it follows that the elements of $[K(N,n), K(M, m+1)]_G = \tilde{H}_G^{m+1}(K(N,n), M)$ that belong to the same $U(L) \times U(R)$-orbit determine G-equivalent G-spaces.

Conversely, suppose that X is a G-simple G-space with only two nontrivial systems of homotopy groups $\underline{\pi}_n(X) = N$, $\underline{\pi}_m(X) = M$ and $f\colon X \to Y$ is a G-equivalence. Naturality of Postnikov systems gives a G-homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ K(N,n) & \xrightarrow{f_n} & K(N,n) \end{array}$$

in which f_n is a G-equivalence.

Let k and k_1 be the equivariant k-invariants of X and Y, respectively. As in the nonequivariant case [**W**], they are related by

$$f_n^*(k_1) = f_*(k) \in \tilde{H}_G^{m+1}(K(N,n), \underline{\pi}_m(Y)),$$

where f_n^* is the isomorphism induced by f_n on Bredon cohomology groups and f_* is induced by the isomorphism $\underline{\pi}_m(X) \to \underline{\pi}_m(Y)$ corresponding to the map f. Identifying $\underline{\pi}_m(X) = \underline{\pi}_m(Y) = M$ and regarding f_* and f_n^* as elements of the groups $U(L)$ and $U(R)$, respectively, we see that k and k_1 belong to the same $U(L) \times U(R)$-orbit.

3. Constructing non-G-equivalent Moore G-spaces of the same type. In this section we shall assume that $G = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. A typical coefficient system for

G can be represented as

$$\begin{array}{ccccc} & \nearrow & M(G/H_1) & \searrow & \\ M(G/G) & \longrightarrow & M(G/H_2) & \longrightarrow & M(G/e) \\ & \searrow & M(G/H_3) & \nearrow & \end{array}$$

where H_1, H_2, H_3 are the proper subgroups of G.

Henceforth, M will denote the rational coefficient system for G given by the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ \mathbf{Q} & \longrightarrow & 0 & \longrightarrow & \mathbf{Q} \\ & \searrow & & \nearrow & \\ & & 0 & & \end{array}$$

where the action on $\mathbf{Q} = M(G/e)$ is trivial. The homological properties of the system M have been examined in [**K**]. We shall use the following result from there.

PROPOSITION 2 [**K**, 5.3.2]. *Let* Ext^i *denote the ith right derived functor of* Hom *in the category of rational coefficient systems. Then*

$$\mathrm{Ext}^i(M,M) = \begin{cases} \mathbf{Q} \oplus \mathbf{Q} & \text{if } i = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

In order to apply Proposition 1, we need the following.

PROPOSITION 3. *Let X be a Moore G-space of type $(M, 2)$. Then X has only two nontrivial systems of homotopy groups $\underline{\pi}_2(X) = \underline{\pi}_3(X) = M$.*

PROOF. We have

$$X^H \simeq \begin{cases} S_0^2 \text{ (the rational 2-sphere)} & \text{if } H = G, e, \\ * \quad \text{otherwise.} \end{cases}$$

Thus the result follows from the fact that

$$\pi_q(S_0^2) = \begin{cases} \mathbf{Q} & \text{if } q = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

See [**G-M**].

We are interested in computing the Bredon cohomology group $\tilde{H}_G^4(K(M,2), M)$ now.

PROPOSITION 4. *There is a functorial short exact sequence of* $\mathbf{Q}$*-vector spaces*

$$0 \to \mathrm{Ext}^2(\underline{\tilde{H}}_2(K(M,2)), M) \to \tilde{H}_G^4(K(M,2), M) \to \mathrm{Hom}(\underline{\tilde{H}}_4(K(M,2)), M) \to 0.$$

PROOF. According to [**B**], there is a functorial universal coefficient spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^p(\underline{\tilde{H}}_q(K(M,2)), M) \Rightarrow \tilde{H}_G^{p+q}(K(M,2), M).$$

From the choice of M it follows that

$$\underline{\tilde{H}}_q(K(M,2)) = \begin{cases} M & \text{for } 0 < q \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Proposition 2, the spectral sequence collapses to the desired short exact sequence.

Let $i_1, i_2 \in [K(M,2), K(M,2)]_G$ be the classes corresponding to $(1,0)$ and $(0,1)$, respectively, under the identification $[K(M,2), K(M,2)]_G = \mathrm{Hom}(M,M) = \mathbf{Q} \oplus \mathbf{Q}$. Let us denote $i_k^2 = i_k \smile i_k$, $k = 1, 2$, where

$$\smile : \tilde{H}^2_G(K(M,2), M) \otimes \tilde{H}^2_G(K(M,2), M) \to \tilde{H}^4_G(K(M,2), M)$$

is the cup-product in Bredon cohomology.

PROPOSITION 5. *For each element*

$$u \in \mathrm{Ext}^2(\underline{\tilde{H}}_2(K(M,2)), M) \subset \tilde{H}^4_G(K(M,2), M)$$

the G-space determined by the equivariant k-invariant $u + i_1^2 + i_2^2$ is a Moore G-space of type $(M,2)$.

PROOF. Any G-map $K(M,2) \to K(M,4)$ representing the element u is null-homotopic when restricted to fixed point spaces of any subgroup H of G. Clearly, the restrictions of i_1^2 and i_2^2 are equal to 0 for $H \neq G$ and e, respectively. On the other hand, it is known that i_1^2 and i_2^2 restricted to $H = G$ and e, respectively, determine spaces homotopy equivalent to S_0^2 (see **[G-M]**).

Let us denote, as in §2, $L = [K(M,4), K(M,4)]_G$, $R = [K(M,2), K(M,2)]_G$. Thus, $L = R = \mathrm{Hom}(M,M) = \mathbf{Q} \oplus \mathbf{Q}$ and $U(L) = U(R) = \mathbf{Q}^* \oplus \mathbf{Q}^*$, where $\mathbf{Q}^*$ is the multiplicative group of nonzero rationals.

We are now going to examine the action of $U(L) \times U(R)$ on $[K(M,2), K(M,4)]_G$ that we have described in §2.

PROPOSITION 6. *Let $((a_1,a_2),(b_1,b_2)) \in U(L) \times U(R)$. Then for each $u \in$ $\mathrm{Ext}^2(\underline{\tilde{H}}_2(K(M,2)), M)$ we have*

$$((a_1,a_2),(b_1,b_2))(u + i_1^2 + i_2^2) = (b_1^{-1}a_2)u + (a_1 b_1^{-2}) i_1^2 + (a_2 b_2^{-2}) i_2^2.$$

PROOF. We may regard the action of $((a_1,a_2),(1,1))$ as a homomorphism induced on the Bredon cohomology groups by a morphism of coefficients, and the action of $((1,1),(b_1,b_2))$ as a homomorphism induced by a G-map $K(M,2) \to K(M,2)$. From this point of view, it is clear that $((a_1,a_2),(b_1,b_2)) i_k^2 = (a_k b_k^{-2}) i_k^2$, $k = 1, 2$.

Let M_1, M_2 be the rational coefficient systems represented by the diagrams

$$\begin{array}{ccccc} & \nearrow & 0 & \searrow & \\ M_1 = \mathbf{Q} & \longrightarrow & 0 & \longrightarrow & 0, \\ & \searrow & 0 & \nearrow & \end{array} \qquad \begin{array}{ccccc} & \nearrow & 0 & \searrow & \\ M_2 = 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{Q}. \\ & \searrow & 0 & \nearrow & \end{array}$$

Then $M = M_1 \oplus M_2$ and it follows from [**T**] that M_2 is projective. By 5.3.1 of [**K**], the 2-term of a minimal projective resolution of M is $M_2 \oplus M_2$ and $\mathrm{Ext}^2(M, M) = \mathrm{Hom}(M_2 \oplus M_2, M)$. Since M_2 is projective, the 2-term of a minimal projective resolution of the morphism $(b_1^{-1}, b_2^{-1})\colon M \to M$ is the same as that of $(b_1^{-1}, 0)$, and the last is easily seen to be multiplication by b_1^{-1}. Hence for $u \in \mathrm{Ext}^2(M, M)$, we get $((1,1),(b_1,b_2))u = b_1^{-1}u$.

We have $\mathrm{Hom}(M_2 \oplus M_2, M) = \mathrm{Hom}(M_2 \oplus M_2, M_2)$. Thus from the equality $\mathrm{Ext}^2(M,M) = \mathrm{Hom}(M_2 \oplus M_2, M)$ it follows that the action of $((a_1,a_2),(1,1))$ on $\mathrm{Ext}^2(M,M)$ is multiplication by a_2.

Since the action of $U(L) \times U(R)$ on $[K(M,2), K(M,4)]_G$ is additive, the proof is complete.

We are now in position to get our main result.

THEOREM. *There exist infinitely many non-G-equivalent Moore G-spaces of type* $(M, 2)$.

PROOF. It follows from Propositions 1, 5 and 6 that any set of elements of $\mathrm{Ext}^2(\underline{\tilde{H}}_2(K(M,2)), M)$, each two elements of which are linearly independent over $\mathbf{Q}$, is in one-to-one correspondence with a set of different G-homotopy types of Moore G-spaces of type $(M,2)$. Since $\mathrm{Ext}^2(\underline{\tilde{H}}_2(K(M,2)), M) = \mathbf{Q} \oplus \mathbf{Q}$, the result follows.

REFERENCES

[**B**] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Math., vol. 34, Springer-Verlag, Berlin and New York, 1967.

[**G-M**] P. A. Griffiths and J. W. Morgan, *Rational homotopy theory and differential forms*, Progress in Math., vol. 16, Birkhäuser, Boston, Mass., 1981.

[**H**] P. J. Hilton, *Homotopy theory and duality*, Gordon and Breach, New York, 1965.

[**K**] P. J. Kahn, *Rational Moore G-spaces*, Trans. Amer. Math. Soc. **298** (1986), 245–271.

[**T**] G. V. Triantafillou, *Equivariant minimal models*, Trans. Amer. Math. Soc. **274** (1982), 509–532.

[**W**] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., no. 61, Springer-Verlag, Berlin and New York, 1978.

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