

NON- G -EQUIVALENT MOORE G -SPACES OF THE SAME TYPE

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ABSTRACT. Let G be a finite group. By a Moore G -space we mean a G -space X such that for each subgroup H of G the fixed point space X^H is a Moore space of type (M_H, n) , where $n > 1$ is a fixed integer and M_H are abelian groups. In this paper it is shown that there exist infinitely many non- G -homotopy equivalent Moore G -spaces of certain given type.

1. Introduction. Let G be a finite group. G -spaces, G -actions, G -maps and G -homotopies considered in this paper will be pointed. We shall work in the category of G -spaces having the G -homotopy type of a G -CW-complex [B], and we make tacit use of the standard strategies for keeping our constructions within this category.

Let O_G be the category of canonical orbits of G , whose objects are the left coset spaces G/H , H a subgroup of G , and whose morphisms are the equivariant (with respect to left translation) maps $G/H \rightarrow G/K$. A *coefficient system* for G is a contravariant functor from O_G into the category of abelian groups. A coefficient system will be called *rational* if its range is the category of \mathbf{Q} -vector spaces. For a G -space X , coefficient systems $\underline{\pi}_n(X)$ and $\tilde{H}_n(X)$ can be defined by $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$, $\tilde{H}_n(X)(G/H) = \tilde{H}_n(X^H)$, where $\tilde{H}_n(\)$ denotes the reduced singular homology with \mathbf{Z} -coefficients.

Let M be a coefficient system for G and $n \geq 2$ an integer.

DEFINITION. A *Moore G -space* of type (M, n) is a G -space X such that each fixed point space X^H is 1-connected and

$$\tilde{H}_q(X) = \begin{cases} M & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Coefficient systems for G and their natural transformations form an abelian category with sufficiently many projectives and injectives [B]. The same holds for rational coefficient systems. Let M be a rational coefficient system and $n \geq 2$ an integer. By a result of P. J. Kahn [K], if $\text{proj. dim } M < n$, then, up to G -equivalence (= G -homotopy equivalence), there exists exactly one Moore G -space of type (M, n) .

In contrast to the non-equivariant case, uniqueness does not hold in general for Moore G -spaces. In [K] there is given an example of a rational coefficient system M and two Moore G -spaces L_1, L_2 of type $(M, 2)$ that are not G -equivalent.

In this paper we extend the above result showing, by quite a different method, that there exist infinitely many non- G -equivalent Moore G -spaces of type $(M, 2)$.

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2. G -spaces with two nontrivial systems of homotopy groups. Suppose that X is a G -simple G -space (i.e. each X^H is connected and simple) having only two nontrivial systems of homotopy groups $\pi_n(X)$ and $\pi_m(X)$ with $n < m$. In such a situation X is determined by its equivariant k -invariant $k(X)$ which lies in the Bredon cohomology group $\tilde{H}_G^{m+1}(K(\pi_n(X), n), \pi_m(X))$, where $K(\pi_n(X), n)$ is an Eilenberg-MacLane G -space of type $(\pi_n(X), n)$ [T]. Let us denote $\pi_n(X) = N$, $\pi_m(X) = M$. The sets $R = [K(N, n), K(N, n)]_G$ and

$$L = [K(M, m + 1), K(M, m + 1)]_G$$

of G -homotopy classes of G -maps are rings with 1, the multiplication being induced by composition of maps. Let $U(L)$, $U(R)$ denote the groups of units (i.e. of classes of G -equivalences) of L and R , respectively. Define an action of the group $U(L) \times U(R)$ on $[K(N, n), K(M, m + 1)]_G$ by the formula $(\alpha, \beta)u = \alpha \circ u \circ \beta^{-1}$, where \circ is induced by composition of maps. Let $V(N, n, M, m + 1)$ denote the orbit space of $[K(N, n), K(M, m + 1)]_G$ under the above $U(L) \times U(R)$ -action.

PROPOSITION 1. *The set $V(N, n, M, m + 1)$ is in one-to-one correspondence with the set of G -homotopy types of G -simple G -spaces X having only two nontrivial systems of homotopy groups $\pi_n(X) = N$, $\pi_m(X) = M$.*

PROOF. From G -homotopy invariance of G -pullbacks, which we get applying formally a dual of Theorem 6.6 of [H] to the G -equivariant case, it follows that the elements of $[K(N, n), K(M, m + 1)]_G = \tilde{H}_G^{m+1}(K(N, n), M)$ that belong to the same $U(L) \times U(R)$ -orbit determine G -equivalent G -spaces.

Conversely, suppose that X is a G -simple G -space with only two nontrivial systems of homotopy groups $\pi_n(X) = N$, $\pi_m(X) = M$ and $f: X \rightarrow Y$ is a G -equivalence. Naturality of Postnikov systems gives a G -homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ K(N, n) & \xrightarrow{f_n} & K(N, n) \end{array}$$

in which f_n is a G -equivalence.

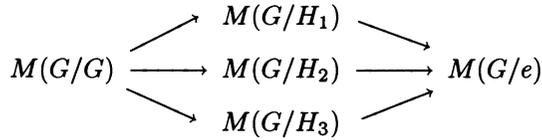
Let k and k_1 be the equivariant k -invariants of X and Y , respectively. As in the nonequivariant case [W], they are related by

$$f_n^*(k_1) = f_*(k) \in \tilde{H}_G^{m+1}(K(N, n), \pi_m(Y)),$$

where f_n^* is the isomorphism induced by f_n on Bredon cohomology groups and f_* is induced by the isomorphism $\pi_m(X) \rightarrow \pi_m(Y)$ corresponding to the map f . Identifying $\pi_m(X) = \pi_m(Y) = M$ and regarding f_* and f_n^* as elements of the groups $U(L)$ and $U(R)$, respectively, we see that k and k_1 belong to the same $U(L) \times U(R)$ -orbit.

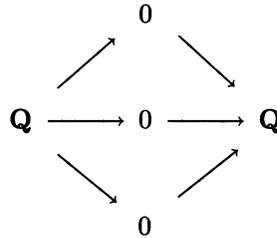
3. Constructing non- G -equivalent Moore G -spaces of the same type. In this section we shall assume that $G = \mathbf{Z}/2 \oplus \mathbf{Z}/2$. A typical coefficient system for

G can be represented as



where H_1, H_2, H_3 are the proper subgroups of G .

Henceforth, M will denote the rational coefficient system for G given by the diagram



where the action on $\mathbf{Q} = M(G/e)$ is trivial. The homological properties of the system M have been examined in [K]. We shall use the following result from there.

PROPOSITION 2 [K, 5.3.2]. *Let Ext^i denote the i th right derived functor of Hom in the category of rational coefficient systems. Then*

$$\text{Ext}^i(M, M) = \begin{cases} \mathbf{Q} \oplus \mathbf{Q} & \text{if } i = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

In order to apply Proposition 1, we need the following.

PROPOSITION 3. *Let X be a Moore G -space of type $(M, 2)$. Then X has only two nontrivial systems of homotopy groups $\pi_2(X) = \pi_3(X) = M$.*

PROOF. We have

$$X^H \simeq \begin{cases} S_0^2 \text{ (the rational 2-sphere)} & \text{if } H = G, e, \\ * & \text{otherwise.} \end{cases}$$

Thus the result follows from the fact that

$$\pi_q(S_0^2) = \begin{cases} \mathbf{Q} & \text{if } q = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

See [G-M].

We are interested in computing the Bredon cohomology group $\tilde{H}_G^4(K(M, 2), M)$ now.

PROPOSITION 4. *There is a functorial short exact sequence of \mathbf{Q} -vector spaces $0 \rightarrow \text{Ext}^2(\tilde{H}_2(K(M, 2)), M) \rightarrow \tilde{H}_G^4(K(M, 2), M) \rightarrow \text{Hom}(\tilde{H}_4(K(M, 2)), M) \rightarrow 0$.*

PROOF. According to [B], there is a functorial universal coefficient spectral sequence with

$$E_2^{p,q} = \text{Ext}^p(\tilde{H}_q(K(M, 2)), M) \Rightarrow \tilde{H}_G^{p+q}(K(M, 2), M).$$

From the choice of M it follows that

$$\tilde{H}_q(K(M, 2)) = \begin{cases} M & \text{for } 0 < q \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by Proposition 2, the spectral sequence collapses to the desired short exact sequence.

Let $i_1, i_2 \in [K(M, 2), K(M, 2)]_G$ be the classes corresponding to $(1, 0)$ and $(0, 1)$, respectively, under the identification $[K(M, 2), K(M, 2)]_G = \text{Hom}(M, M) = \mathbf{Q} \oplus \mathbf{Q}$. Let us denote $i_k^2 = i_k \smile i_k, k = 1, 2$, where

$$\smile: \tilde{H}_G^2(K(M, 2), M) \otimes \tilde{H}_G^2(K(M, 2), M) \rightarrow \tilde{H}_G^4(K(M, 2), M)$$

is the cup-product in Bredon cohomology.

PROPOSITION 5. For each element

$$u \in \text{Ext}^2(\tilde{H}_2(K(M, 2)), M) \subset \tilde{H}_G^4(K(M, 2), M)$$

the G -space determined by the equivariant k -invariant $u + i_1^2 + i_2^2$ is a Moore G -space of type $(M, 2)$.

PROOF. Any G -map $K(M, 2) \rightarrow K(M, 4)$ representing the element u is null-homotopic when restricted to fixed point spaces of any subgroup H of G . Clearly, the restrictions of i_1^2 and i_2^2 are equal to 0 for $H \neq G$ and e , respectively. On the other hand, it is known that i_1^2 and i_2^2 restricted to $H = G$ and e , respectively, determine spaces homotopy equivalent to S_0^2 (see [G-M]).

Let us denote, as in §2, $L = [K(M, 4), K(M, 4)]_G, R = [K(M, 2), K(M, 2)]_G$. Thus, $L = R = \text{Hom}(M, M) = \mathbf{Q} \oplus \mathbf{Q}$ and $U(L) = U(R) = \mathbf{Q}^* \oplus \mathbf{Q}^*$, where \mathbf{Q}^* is the multiplicative group of nonzero rationals.

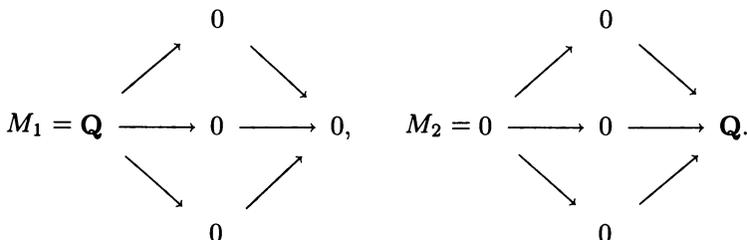
We are now going to examine the action of $U(L) \times U(R)$ on $[K(M, 2), K(M, 4)]_G$ that we have described in §2.

PROPOSITION 6. Let $((a_1, a_2), (b_1, b_2)) \in U(L) \times U(R)$. Then for each $u \in \text{Ext}^2(\tilde{H}_2(K(M, 2)), M)$ we have

$$((a_1, a_2), (b_1, b_2))(u + i_1^2 + i_2^2) = (b_1^{-1}a_2)u + (a_1b_1^{-2})i_1^2 + (a_2b_2^{-2})i_2^2.$$

PROOF. We may regard the action of $((a_1, a_2), (1, 1))$ as a homomorphism induced on the Bredon cohomology groups by a morphism of coefficients, and the action of $((1, 1), (b_1, b_2))$ as a homomorphism induced by a G -map $K(M, 2) \rightarrow K(M, 2)$. From this point of view, it is clear that $((a_1, a_2), (b_1, b_2))i_k^2 = (a_k b_k^{-2})i_k^2, k = 1, 2$.

Let M_1, M_2 be the rational coefficient systems represented by the diagrams



Then $M = M_1 \oplus M_2$ and it follows from [T] that M_2 is projective. By 5.3.1 of [K], the 2-term of a minimal projective resolution of M is $M_2 \oplus M_2$ and $\text{Ext}^2(M, M) = \text{Hom}(M_2 \oplus M_2, M)$. Since M_2 is projective, the 2-term of a minimal projective resolution of the morphism $(b_1^{-1}, b_2^{-1}): M \rightarrow M$ is the same as that of $(b_1^{-1}, 0)$, and the last is easily seen to be multiplication by b_1^{-1} . Hence for $u \in \text{Ext}^2(M, M)$, we get $((1, 1), (b_1, b_2))u = b_1^{-1}u$.

We have $\text{Hom}(M_2 \oplus M_2, M) = \text{Hom}(M_2 \oplus M_2, M_2)$. Thus from the equality $\text{Ext}^2(M, M) = \text{Hom}(M_2 \oplus M_2, M)$ it follows that the action of $((a_1, a_2), (1, 1))$ on $\text{Ext}^2(M, M)$ is multiplication by a_2 .

Since the action of $U(L) \times U(R)$ on $[K(M, 2), K(M, 4)]_G$ is additive, the proof is complete.

We are now in position to get our main result.

THEOREM. *There exist infinitely many non-G-equivalent Moore G-spaces of type $(M, 2)$.*

PROOF. It follows from Propositions 1, 5 and 6 that any set of elements of $\text{Ext}^2(\tilde{H}_2(K(M, 2)), M)$, each two elements of which are linearly independent over \mathbf{Q} , is in one-to-one correspondence with a set of different G -homotopy types of Moore G -spaces of type $(M, 2)$. Since $\text{Ext}^2(\tilde{H}_2(K(M, 2)), M) = \mathbf{Q} \oplus \mathbf{Q}$, the result follows.

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