SIMULTANEOUS SYSTEMS OF REPRESENTATIVES FOR FAMILIES OF FINITE SETS

MELVYN B. NATHANSON

(Communicated by Andrew Odlyzko)

ABSTRACT. Let $h \geq 2$ and $k \geq 1$. Then there exists a real number $\lambda = \lambda(h,k) \in (0,1)$ such that, if $\mathcal{S} = \{S_i\}_{i=1}^{s}$ and $\mathcal{T} = \{T_j\}_{j=1}^{t}$ are families of nonempty, pairwise disjoint sets with $|S_i| \leq h$ and $|T_j| \leq k$ and $S_i \nsubseteq T_j$ for all $i$ and $j$, then $N(\mathcal{S},\mathcal{T}) \leq h^s \lambda^t$, where $N(\mathcal{S},\mathcal{T})$ is the number of sets $X$ such that $X$ is a minimal system of representatives for $\mathcal{S}$ and $X$ is simultaneously a system of representatives for $\mathcal{T}$. A conjecture about the best possible value of the constant $\lambda(h,k)$ is proved in the case $h > k$. The necessity of the disjointness conditions for the families $\mathcal{S}$ and $\mathcal{T}$ is also demonstrated.

Let $\mathcal{S} = \{S_i\}_{i=1}^{s}$ and $\mathcal{T} = \{T_j\}_{j=1}^{t}$ be two families of nonempty sets. The set $X$ is a system of representatives for $\mathcal{S}$ if $X \cap S_i \neq \emptyset$ for all $i \in I$. If $X$ is a system of representatives for $\mathcal{S}$ but no proper subset of $X$ is a system of representatives for $\mathcal{S}$, then $X$ is a minimal system of representatives for $\mathcal{S}$.

Let $\mathcal{M}(\mathcal{S})$ denote the number of minimal systems of representatives for $\mathcal{S}$. If the sets $S_i$ are pairwise disjoint, and if $|S_i| \leq h$, where $|S|$ denotes the cardinality of the set $S$, then $\mathcal{M}(\mathcal{S}) = \prod_{i=1}^{s} |S_i| \leq h^s$.

Let $N(\mathcal{S},\mathcal{T})$ denote the number of sets $X$ such that $X$ is a minimal system of representatives for $\mathcal{S}$ and $X$ is simultaneously a system of representatives for $\mathcal{T}$. At the Third International Conference on Combinatorial Mathematics in New York in June, 1985, the author stated the following two conjectures about the number $N(\mathcal{S},\mathcal{T})$ in the case of two finite families of pairwise disjoint finite sets.

CONJECTURE 1. Let $h \geq 2$ and $k \geq 1$. There exists a real number $\lambda = \lambda(h,k) \in (0,1)$ with the following property:

(i) Let $\mathcal{S} = \{S_i\}_{i=1}^{s}$ be a family of $s$ nonempty, pairwise disjoint sets $S_i$ with $|S_i| \leq h$ for all $i$.

(ii) Let $\mathcal{T} = \{T_j\}_{j=1}^{t}$ be a family of $t$ nonempty, pairwise disjoint sets $T_j$ with $|T_j| \leq k$ for all $j$.

(iii) Suppose $S_i \nsubseteq T_j$ for all $i$ and $j$. Then

\[ N(\mathcal{S},\mathcal{T}) \leq h^s \lambda^t. \]

CONJECTURE 2. Let $h \geq 2$ and $k \geq 1$. Let $k = q(h-1) + r$, where $0 \leq r \leq h-2$. Define $\lambda^*(h,k) \in (0,1)$ by

\[ \lambda^*(h,k) = 1 - (h - r)/h^{q+1}. \]

Received by the editors October 23, 1986 and, in revised form, June 15, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 05A05, 11B13, 11B75.

Key words and phrases. Systems of representatives, additive bases.

©1988 American Mathematical Society

0002-9939/88 $1.00 + $.25 per page

1322
Then \( \lambda^*(h, k) \) is the smallest value of \( \lambda \) that satisfies inequality (1) of Conjecture 1.

Note that Conjectures 1 and 2 imply that

\[
N(S^*, T^*) < h^s - 2t(h^2 - h + 1)^t
\]

if \( h = k \), and that

\[
N(S^*, T^*) < h^s - t_k^t
\]

if \( h > k \).

Erdös and Nathanson [1] proved Conjectures 1 and 2 in the special case \( h = k - 2 \), and they applied this result in additive number theory to obtain a sufficient condition for an asymptotic basis of order 2 to contain a minimal asymptotic basis of order 2. Conjectures 1 and 2 should have important applications to the study of additive bases of orders greater than 2.

**Theorem 1.** Let \( h \geq 2 \) and \( k \geq 1 \). Let \( \lambda \in (0, 1) \). If \( N(S^*, T^*) < h^s \lambda^t \) for all families \( S^* \) and \( T^* \) that satisfy conditions (i), (ii), and (iii) of Conjecture 1, then \( \lambda \geq \lambda^*(h, k) \).

**Proof.** Let \( d_1, \ldots, d_s \) be positive integers such that \( d_i \leq h - 1 \) for all \( i \) and \( d_1 + \cdots + d_s \leq k \). Let \( S_1, \ldots, S_s \) be pairwise disjoint sets with \( |S_i| = h \) for all \( i \). Let \( T \) be a set such that \( |T| = k \) and \( |T \cap S_i| = d_i \) for \( i = 1, \ldots, s \). Let \( S = \{S_i\}_{i=1}^s \) and \( T = \{T\} \). Then

\[
N(S, T) = h^s - \prod_{i=1}^s (h - d_i) = h^s \left( 1 - \prod_{i=1}^s \left( 1 - \frac{d_i}{h} \right) \right)
\]

and so

\[
(2) \quad \lambda \geq 1 - \prod_{i=1}^s \left( 1 - \frac{d_i}{h} \right).
\]

Let \( k = q(h - 1) + r \), where \( q = \lfloor k/(h - 1) \rfloor \) and \( 0 \leq r \leq h - 2 \). Let \( s = q + 1 \). Let \( d_i = h - 1 \) for \( i = 1, \ldots, s - 1 \) and \( d_s = r \). Then the right side of inequality (2) is \( \lambda^*(h, k) \). This proves the theorem.

**Definition.** Let \( S \) and \( T \) be families of sets. Then \( T \) is independent with respect to \( S \) if to each set \( S \in S \) there is at most one set \( T \in T \) such that

\[
S \cap T \neq \emptyset.
\]

**Lemma.** Let \( S \) and \( T \) satisfy conditions (i) and (ii) of Conjecture 1. If \( T' \) is a maximal subfamily of \( T \) that is independent with respect to \( S \), and if \( t' = |T'| \), then \( t' \geq t/(h - 1 + 1) \).

**Proof.** Let \( T' = \{T_j\}_{j=1}^{t'} \) be a maximal subfamily of \( T \). For each \( j \in [1, t'] \), there is a set \( I(j) \subseteq [1, s] \) such that \( |I(j)| \leq k \) and \( S_i \cap T_j \neq \emptyset \) if and only if \( i \in I(j) \).

There are \( t - t' \) sets belonging to \( T \setminus T' \). Each of these sets must intersect some set \( S_i \), where \( i \in I(j) \) for some \( j = 1, \ldots, t' \). Since each of these sets \( S_i \) also
intersects some set $T_j$ belonging to the family $\mathcal{F}'$, and since the sets in $\mathcal{F}$ are pairwise disjoint, it follows that

$$t - t' \leq \sum_{j=1}^{t'} \sum_{i \in I(j)} (|S_i| - 1) \leq (h - 1) \sum_{j=1}^{t'} |I(j)| \leq (h - 1)kt'. $$

This proves the Lemma.

The following result implies Conjecture 1.

**THEOREM 2.** Let $h \geq 2$ and $k \geq 1$. There exists a real number $\lambda = \lambda(h, k) \in (0, 1)$ with the following property:

(i) Let $\mathcal{S} = \{S_i\}_{i=1}^{s}$ be a family of $s$ nonempty, pairwise disjoint sets $S_i$ with $|S_i| \leq h$ for all $i$.

(ii) Let $\mathcal{T} = \{T_j\}_{j=1}^{t}$ be a family of $t$ nonempty, pairwise disjoint sets $T_j$ with $|T_j| \leq k$ for all $j$.

(iii) Suppose $S_i \not\subseteq T_j$ for all $i$ and $j$. Then

$$N(\mathcal{S}, \mathcal{T}) \leq M(\mathcal{S})\lambda^t.$$  

**PROOF.** Define $\mu = 1 - 1/hk$ and $\lambda \equiv \mu^{1/(h-1)k+1}$. Then $\mu < \lambda$. It suffices to prove that if $\mathcal{T}$ is independent with respect to $\mathcal{S}$, then $N(\mathcal{S}, \mathcal{T}) \leq M(\mathcal{S})\mu^t$. The reason is as follows: Let $\mathcal{F}'$ be a maximal subfamily of $\mathcal{F}$ that is independent with respect to $\mathcal{S}$, and let $t' = |\mathcal{F}'|$. Then the Lemma implies that

$$N(\mathcal{S}, \mathcal{F}') \leq N(\mathcal{S}, \mathcal{T}) \leq M(\mathcal{S})\mu^t \leq M(\mathcal{S})\lambda^t.$$

Now assume that $\mathcal{T}$ is independent with respect to $\mathcal{S}$. Note that the result is trivial if $\mathcal{T} = \emptyset$, that is, if $t = 0$. Suppose that $t = 1$ and $\mathcal{T} = \{T\}$. Let $d_i = |S_i \cap T|$. Since $S_i \not\subseteq T$, it follows that $d_i \leq |S_i| - 1$. Reorder the sets $S_i$ so that $d_i > 0$ for $i = 1, \ldots, s'$ and $d_i = 0$ for $i = s' + 1, \ldots, s$. Then $s' \leq |T| \leq k$ and

$$N(\mathcal{S}, \mathcal{T}) = \prod_{i=1}^{s} |S_i| - \prod_{i=s'+1}^{s} |S_i| \prod_{i=1}^{s'} (|S_i| - d_i)$$

$$= M(\mathcal{S}) \left(1 - \prod_{i=1}^{s'} \left(1 - \frac{d_i}{|S_i|}\right)\right)$$

$$\leq M(\mathcal{S}) \left(1 - \prod_{i=1}^{s'} \left(\frac{1}{|S_i|}\right)\right)$$

$$\leq M(\mathcal{S})(1 - h^{-k}) = M(\mathcal{S})\mu.$$

Now let $\mathcal{F}$ be any family that is independent with respect to $\mathcal{S}$. Since each set $S_i \in \mathcal{S}$ intersects at most one set $T_j \in \mathcal{F}$, we can partition the family $\mathcal{F}$ into $t$ families $\mathcal{F}_j$ such that if $S \in \mathcal{F}$ and if $S \cap T_j \neq \emptyset$, then $S \in \mathcal{F}_j$. Let $\mathcal{F}_j = \{T_j\}$. Then

$$N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}, \mathcal{F}_1) \cdots N(\mathcal{S}, \mathcal{F}_t) \leq M(\mathcal{F}_1) \cdots M(\mathcal{F}_t)\mu^t = M(\mathcal{S})\mu^t.$$

This completes the proof of the Theorem.

The following result proves Conjecture 2 in the case $h > k$. 
**THEOREM 3.** Let \( h > k \geq 1 \). Let the families \( \mathcal{S} \) and \( \mathcal{T} \) satisfy conditions (i), (ii), and (iii) of Theorem 2. Then \( N(\mathcal{S}, \mathcal{T}) \leq h^{s-t}k^t \).

**PROOF.** Let \( X' \) be a minimal system of representatives for \( \mathcal{S} \), and let \( L(X') \) denote the number of minimal systems of representatives for \( \mathcal{S} \) that contain \( X' \). If the elements of \( X' \) do not belong to distinct sets \( S_i \) in the family \( \mathcal{S} \), then \( L(X') = 0 \). If the \( t \) elements of \( X' \) do belong to \( t \) distinct sets in \( \mathcal{S} \), then there are at most \( h^{s-t} \) ways to choose a set \( X'' \) of representatives from the remaining \( s-t \) sets in \( \mathcal{S} \) so that \( X = X' \cup X'' \) will be a minimal system of representatives for \( \mathcal{S} \). Moreover, there are at most \( k^t \) minimal systems \( X' \) of representatives for \( \mathcal{T} \). Since every set \( X \) counted in \( N(\mathcal{S}, \mathcal{T}) \) contains some set \( X' \), it follows that

\[
N(\mathcal{S}, \mathcal{T}) \leq \sum_{X'} L(X') \leq h^{s-t}k^t.
\]

This proves the theorem.

The following two results show the necessity in Theorem 2 that the families \( \mathcal{S} \) and \( \mathcal{T} \) consist of pairwise disjoint sets.

**THEOREM 4.** Let \( h \geq 2 \) and \( k \geq 1 \). Let \( \lambda \in (0, 1) \). Then there is a family \( \mathcal{S} \) of nonempty, distinct sets \( S \) with \( |S| = h \), and a set \( T \) with \( |T| = k \) and \( S \notin T \) such that, if \( T = \{T\} \), then \( N(\mathcal{S}, \mathcal{T}) > M(\mathcal{S})\lambda \).

**PROOF.** Choose \( n > (h - 1)/(1 - \lambda) \). Let \( S^* \) be a set with \( |S^*| = n \), and let \( \mathcal{S} \) be the family of all \( h \)-element subsets of \( S^* \). Then

\[
|\mathcal{S}| = s = \binom{n}{h} \quad \text{and} \quad M(\mathcal{S}) = \binom{n}{h}.
\]

Let \( T \) be a set such that \( |T| = k \) and \( |T \cap S^*| = 1 \). Then

\[
N(\mathcal{S}, \mathcal{T}) = \binom{n-1}{h-1} = M(\mathcal{S}) \left(1 - \frac{h-1}{n}\right) > M(\mathcal{S})\lambda.
\]

**THEOREM 5.** Let \( h \geq 2 \) and \( k \geq 2 \). For any \( \lambda \in (0, 1) \) and \( \varepsilon > 0 \) there exist families \( \mathcal{S} = \{S_i\}_{i=1}^s \) and \( \mathcal{T} = \{T_j\}_{j=1}^t \) with the following properties:

(i) The \( s \) sets \( S_i \) are pairwise disjoint and \( |S_i| = h \) for all \( i \).

(ii) The \( t \) sets \( T_j \) are distinct and \( |T_j| = k \) for all \( j \).

(iii) \( S_i \notin T_j \) for all \( i \) and \( j \).

(iv) \( t < \varepsilon s \).

(v) \( N(\mathcal{S}, \mathcal{T}) > h^\lambda \varepsilon^2 \).

**PROOF.** Let \( n = \min(h, k) \). Then \( n \geq 2 \) and

\[
\frac{r(h-1)}{n-1} - r > cr^n
\]

for some \( c > 0 \) and all \( r \geq r_0 \). Choose \( r \) sufficiently large so that \( r \geq r_0 \) and \( h^\lambda cr^n < 1 \). Let \( \mathcal{S}' = \{S_i\}_{i=1}^r \) be a family of \( r \) pairwise disjoint sets \( S_i \) with \( |S_i| = h \) for \( i = 1, \ldots, r \). Choose \( x_i \in S_i \). Then \( X' = \{x_1, \ldots, x_r\} \) is a minimal system of representatives for \( \mathcal{S}' \).

Let \( S = \bigcup_{i=1}^r S_i \). Let \( V \) be a set such that \( V \cap S = \emptyset \) and \( |V| = k - n \). Let \( \mathcal{T} \) consist of all sets \( T = \{x_i\} \cup T' \cup V \), where \( x_i \in X' \) and \( T' \subseteq S \setminus X' \) satisfies
\[ |T'| = n - 1 \] and \[ T' \neq S_i \backslash \{x_i\} \]. Then \[ S_i \notin T \] for all \( S_i \in \mathcal{S} \) and \( T \in \mathcal{T} \). Since \[ |S\backslash X'| = r(h - 1) \], the family \( \mathcal{T} \) consists of

\[
t \geq r \left( \frac{r(h - 1)}{n - 1} \right) - r > cr^n
\]
distinct sets of cardinality \( k \).

Clearly, \( X' \) is a minimal system of representatives for \( \mathcal{S}' \) and a system of representatives for \( \mathcal{S} \), but

\[
h^r \lambda^t < h^r \lambda^{cr^n} < 1 \leq N(\mathcal{S}', \mathcal{T}).
\]

Choose \( s > r \) so large that \( t < \varepsilon s \). Let \( S_{r+1}, \ldots, S_s \) be pairwise disjoint sets of cardinality \( h \) that are also disjoint from the sets in \( \mathcal{S}' \) and \( \mathcal{T} \). Let

\[
\mathcal{S} = \mathcal{S}' \cup \{S_i\}_{i=r+1}^s = \{S_i\}_{i=1}^s.
\]

There are \( h^{s-r} \) minimal systems of representatives for \( \mathcal{S} \) that contain \( X' \), and so

\[
h^s \lambda^t = h^{s-r} h^r \lambda^t < h^{s-r} \leq N(\mathcal{S}, \mathcal{T}).
\]

The families \( \mathcal{S} \) and \( \mathcal{T} \) satisfy conditions (i)-(v) of the theorem.

**REMARKS.** Xing-De Jia has recently given complete proofs of Conjectures 1 and 2.

I wish to thank the referee for numerous comments that greatly improved this paper.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, LEHMAN COLLEGE (CUNY), BRONX, NEW YORK 10468**