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SIMULTANEOUS SYSTEMS OF REPRESENTATIVES FOR FAMILIES OF FINITE SETS

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ABSTRACT. Let $h \geq 2$ and $k \geq 1$. Then there exists a real number $\lambda = \lambda(h,k) \in (0,1)$ such that, if $\mathscr{S} = \{S_i\}_{i=1}^s$ and $\mathscr{T} = \{T_j\}_{j=1}^t$ are families of nonempty, pairwise disjoint sets with $|S_i| \leq h$ and $|T_j| \leq k$ and $S_i \not\subseteq T_j$ for all i and j, then $N(\mathscr{S},\mathscr{T}) \leq h^s\lambda^t$, where $N(\mathscr{S},\mathscr{T})$ is the number of sets X such that X is a minimal system of representatives for $\mathscr{S}$ and X is simultaneously a system of representatives for $\mathscr{T}$. A conjecture about the best possible value of the constant $\lambda(h,k)$ is proved in the case $h > k$. The necessity of the disjointness conditions for the families $\mathscr{S}$ and $\mathscr{T}$ is also demonstrated.

Let $\mathscr{S} = \{S_i\}_{i=1}^s$ and $\mathscr{T} = \{T\}_{j=1}^t$ be two families of nonempty sets. The set X is a *system of representatives* for $\mathscr{S}$ if $X \cap S_i \neq \varnothing$ for all $i \in I$. If X is a system of representatives for $\mathscr{S}$ but no proper subset of X is a system of representatives for $\mathscr{S}$, then X is a *minimal system of representatives* for $\mathscr{S}$.

Let $M(\mathscr{S})$ denote the number of minimal systems of representatives for $\mathscr{S}$. If the sets S_i are pairwise disjoint, and if $|S_i| \leq h$, where $|S|$ denotes the cardinality of the set S, then $M(\mathscr{S}) = \prod_{i=1}^s |S_i| \leq h^s$.

Let $N(\mathscr{S},\mathscr{T})$ denote the number of sets X such that X is a minimal system of representatives for $\mathscr{S}$ and X is simultaneously a system of representatives for $\mathscr{T}$. At the Third International Conference on Combinatorial Mathematics in New York in June, 1985, the author stated the following two conjectures about the number $N(\mathscr{S},\mathscr{T})$ in the case of two finite families of pairwise disjoint finite sets.

CONJECTURE 1. *Let $h \geq 2$ and $k \geq 1$. There exists a real number $\lambda = \lambda(h,k) \in (0,1)$ with the following property:*

(i) *Let $\mathscr{S} = \{S_i\}_{i=1}^s$ be a family of s nonempty, pairwise disjoint sets S_i with $|S_i| \leq h$ for all i.*

(ii) *Let $\mathscr{T} = \{T_j\}_{j=1}^t$ be a family of t nonempty, pairwise disjoint sets T_j with $|T_j| \leq k$ for all j.*

(iii) *Suppose $S_i \not\subseteq T_j$ for all i and j. Then*

$$N(\mathscr{S},\mathscr{T}) \leq h^s\lambda^t. \tag{1}$$

CONJECTURE 2. *Let $h \geq 2$ and $k \geq 1$. Let $k = q(h-1)+r$, where $0 \leq r \leq h-2$. Define $\lambda^*(h,k) \in (0,1)$ by*

$$\lambda^*(h,k) = 1 - (h-r)/h^{q+1}.$$

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Then $\lambda^*(h,k)$ *is the smallest value of* λ *that satisfies inequality* (1) *of Conjecture* 1.

Note that Conjectures 1 and 2 imply that

$$N(\mathcal{S},\mathcal{T}) \le h^{s-2t}(h^2-h+1)^t$$

if $h=k$, and that

$$N(\mathcal{S},\mathcal{T}) \le h^{s-t}k^t$$

if $h>k$.

Erdös and Nathanson [**1**] proved Conjectures 1 and 2 in the special case $h = k = 2$, and they applied this result in additive number theory to obtain a sufficient condition for an asymptotic basis of order 2 to contain a minimal asymptotic basis of order 2. Conjectures 1 and 2 should have important applications to the study of additive bases of orders greater than 2.

THEOREM 1. *Let* $h\ge 2$ *and* $k \ge 1$. *Let* $\lambda\in(0,1)$. *If* $N(\mathcal{S},\mathcal{T})\le h^s\lambda^t$ *for all families* $\mathcal{S}$ *and* $\mathcal{T}$ *that satisfy conditions* (i), (ii), *and* (iii) *of Conjecture* 1, *then* $\lambda\ge\lambda^*(h,k)$.

PROOF. Let $d_1,\ldots,d_s$ be positive integers such that $d_i\le h-1$ for all i and $d_1+\cdots+d_s\le k$. Let $S_1,\ldots,S_s$ be pairwise disjoint sets with $|S_i|=h$ for all i. Let T be a set such that $|T|=k$ and $|T\cap S_i|=d_i$ for $i=1,\ldots,s$. Let $\mathcal{S}=\{S_i\}_{i=1}^s$ and $\mathcal{T}=\{T\}$. Then

$$N(\mathcal{S},\mathcal{T}) = h^s-\prod_{i=1}^s(h-d_i) = h^s\left(1-\prod_{i=1}^s\left(1-\frac{d_i}{h}\right)\right)$$

and so

$$\lambda\ge 1-\prod_{i=1}^s\left(1-\frac{d_i}{h}\right). \tag{2}$$

Let $k=q(h-1)+r$, where $q=[k/(h-1)]$ and $0\le r\le h-2$. Let $s=q+1$. Let $d_i=h-1$ for $i=1,\ldots,s-1$ and $d_s=r$. Then the right side of inequality (2) is $\lambda^*(h,k)$. This proves the theorem.

DEFINITION. Let $\mathcal{S}$ and $\mathcal{T}$ be families of sets. Then $\mathcal{T}$ *is independent with respect to* $\mathcal{S}$ if to each set $S\in\mathcal{S}$ there is at most one set $T\in\mathcal{T}$ such that $S\cap T\ne\varnothing$.

LEMMA. *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *satisfy conditions* (i) *and* (ii) *of Conjecture* 1. *If* $\mathcal{T}'$ *is a maximal subfamily of* $\mathcal{T}$ *that is independent with respect to* $\mathcal{S}$, *and if* $t'=|\mathcal{T}'|$, *then* $t'\ge t/((h-1)k+1)$.

PROOF. Let $\mathcal{T}'=\{T_j\}_{j=1}^{t'}$ be a maximal subfamily of $\mathcal{T}$. For each $j\in[1,t']$, there is a set $I(j)\subseteq[1,s]$ such that $|I(j)|\le k$ and $S_i\cap T_j\ne\varnothing$ if and only if $i\in I(j)$.

There are $t-t'$ sets belonging to $\mathcal{T}\backslash\mathcal{T}'$. Each of these sets must intersect some set S_i, where $i\in I(j)$ for some $j=1,\ldots,t'$. Since each of these sets S_i also intersects some set T_j belonging to the family $\mathcal{T}'$, and since the sets in $\mathcal{T}$ are pairwise disjoint, it follows that

$$t - t' \le \sum_{j=1}^{t'} \sum_{i \in I(j)} (|S_i| - 1) \le (h-1) \sum_{j=1}^{t'} |I(j)| \le (h-1)kt'.$$

This proves the Lemma.

The following result implies Conjecture 1.

THEOREM 2. *Let $h \ge 2$ and $k \ge 1$. There exists a real number $\lambda = \lambda(h,k) \in (0,1)$ with the following property:*

(i) *Let $\mathcal{S} = \{S_i\}_{i=1}^{s}$ be a family of s nonempty, pairwise disjoint sets S_i with $|S_i| \le h$ for all i.*

(ii) *Let $\mathcal{T} = \{T_j\}_{j=1}^{t}$ be a family of t nonempty, pairwise disjoint sets T_j with $|T_j| \le k$ for all j.*

(iii) *Suppose $S_i \not\subseteq T_j$ for all i and j. Then*

$$N(\mathcal{S}, \mathcal{T}) \le M(\mathcal{S}) \lambda^t. \tag{3}$$

PROOF. Define $\mu = 1 - 1/h^k$ and $\lambda \equiv \mu^{1/((h-1)k+1)}$. Then $\mu < \lambda$. It suffices to prove that if $\mathcal{T}$ is independent with respect to $\mathcal{S}$, then $N(\mathcal{S}, \mathcal{T}) \le M(\mathcal{S})\mu^t$. The reason is as follows: Let $\mathcal{T}'$ be a maximal subfamily of $\mathcal{T}$ that is independent with respect to $\mathcal{S}$, and let $t' = |\mathcal{T}'|$. Then the Lemma implies that

$$N(\mathcal{S}, \mathcal{T}) \le N(\mathcal{S}, \mathcal{T}') \le M(\mathcal{S})\mu^{t'} \le M(\mathcal{S})\lambda^t.$$

Now assume that $\mathcal{T}$ is independent with respect to $\mathcal{S}$. Note that the result is trivial if $\mathcal{T} = \varnothing$, that is, if $t = 0$. Suppose that $t = 1$ and $\mathcal{T} = \{T\}$. Let $d_i = |S_i \cap T|$. Since $S_i \not\subseteq T$, it follows that $d_i \le |S_i| - 1$. Reorder the sets S_i so that $d_i > 0$ for $i = 1, \ldots, s'$ and $d_i = 0$ for $i = s'+1, \ldots, s$. Then $s' \le |T| \le k$ and

$$\begin{aligned}
N(\mathcal{S}, \mathcal{T}) &= \prod_{i=1}^{s} |S_i| - \prod_{i=s'+1}^{s} |S_i| \prod_{i=1}^{s'} (|S_i| - d_i) \\
&= M(\mathcal{S}) \left(1 - \prod_{i=1}^{s'} \left(1 - \frac{d_i}{|S_i|}\right)\right) \\
&\le M(\mathcal{S}) \left(1 - \prod_{i=1}^{s'} \left(\frac{1}{|S_i|}\right)\right) \\
&\le M(\mathcal{S})(1 - h^{-k}) = M(\mathcal{S})\mu.
\end{aligned}$$

Now let $\mathcal{T}$ be any family that is independent with respect to $\mathcal{S}$. Since each set $S_i \in \mathcal{S}$ intersects at most one set $T_j \in \mathcal{T}$, we can partition the family $\mathcal{S}$ into t families $\mathcal{S}_j$ such that if $S \in \mathcal{S}$ and if $S \cap T_j \ne \varnothing$, then $S \in \mathcal{S}_j$. Let $\mathcal{T}_j = \{T_j\}$. Then

$$N(\mathcal{S}, \mathcal{T}) = N(\mathcal{S}_1, \mathcal{T}_1) \cdots N(\mathcal{S}_t, \mathcal{T}_t) \le M(\mathcal{S}_1) \cdots M(\mathcal{S}_t)\mu^t = M(\mathcal{S})\mu^t.$$

This completes the proof of the Theorem.

The following result proves Conjecture 2 in the case $h > k$.

THEOREM 3. *Let $h > k \geq 1$. Let the families $\mathscr{S}$ and $\mathscr{T}$ satisfy conditions* (i), (ii), *and* (iii) *of Theorem* 2. *Then $N(\mathscr{S},\mathscr{T}) \leq h^{s-t}k^t$.*

PROOF. Let X' be a minimal system of representatives for $\mathscr{T}$, and let $L(X')$ denote the number of minimal systems of representatives for $\mathscr{S}$ that contain X'. If the elements of X' do not belong to distinct sets S_i in the family $\mathscr{S}$, then $L(X') = 0$. If the t elements of X' do belong to t distinct sets in $\mathscr{S}$, then there are at most h^{s-t} ways to choose a set X'' of representatives from the remaining $s - t$ sets in $\mathscr{S}$ so that $X = X' \cup X''$ will be a minimal system of representatives for $\mathscr{S}$. Moreover, there are at most k^t minimal systems X' of representatives for $\mathscr{T}$. Since every set X counted in $N(\mathscr{S},\mathscr{T})$ contains some set X', it follows that

$$N(\mathscr{S},\mathscr{T}) \leq \sum_{X'} L(X') \leq h^{s-t}k^t.$$

This proves the theorem.

The following two results show the necessity in Theorem 2 that the families $\mathscr{S}$ and $\mathscr{T}$ consist of pairwise disjoint sets.

THEOREM 4. *Let $h \geq 2$ and $k \geq 1$. Let $\lambda \in (0,1)$. Then there is a family $\mathscr{S}$ of nonempty, distinct sets S with $|S| = h$, and a set T with $|T| = k$ and $S \not\subseteq T$ such that, if $\mathscr{T} = \{T\}$, then $N(\mathscr{S},\mathscr{T}) > M(\mathscr{S})\lambda$.*

PROOF. Choose $n > (h-1)/(1-\lambda)$. Let S^* be a set with $|S^*| = n$, and let $\mathscr{S}$ be the family of all h-element subsets of S^*. Then

$$|\mathscr{S}| = s = \binom{n}{h} \quad \text{and} \quad M(\mathscr{S}) = \binom{n}{h-1}.$$

Let T be a set such that $|T| = k$ and $|T \cap S^*| = 1$. Then

$$N(\mathscr{S},\mathscr{T}) = \binom{n-1}{h-1} = M(\mathscr{S})\left(1 - \frac{h-1}{n}\right) > M(\mathscr{S})\lambda.$$

THEOREM 5. *Let $h \geq 2$ and $k \geq 2$. For any $\lambda \in (0,1)$ and $\varepsilon > 0$ there exist families $\mathscr{S} = \{S_i\}_{i=1}^s$ and $\mathscr{T} = \{T_j\}_{j=1}^t$ with the following properties:*
(i) *The s sets S_i are pairwise disjoint and $|S_i| = h$ for all i.*
(ii) *The t sets T_j are distinct and $|T_j| = k$ for all j.*
(iii) *$S_i \not\subseteq T_j$ for all i and j.*
(iv) *$t < \varepsilon s$.*
(v) *$N(\mathscr{S},\mathscr{T}) > h^s\lambda^t$.*

PROOF. Let $n = \min(h,k)$. Then $n \geq 2$ and

$$r\binom{r(h-1)}{n-1} - r > cr^n$$

for some $c > 0$ and all $r \geq r_0$. Choose r sufficiently large so that $r \geq r_0$ and $h^r\lambda^{cr^n} < 1$. Let $\mathscr{S}' = \{S_i\}_{i=1}^r$ be a family of r pairwise disjoint sets S_i with $|S_i| = h$ for $i = 1, \ldots, r$. Choose $x_i \in S_i$. Then $X' = \{x_1, \ldots, x_r\}$ is a minimal system of representatives for $\mathscr{S}'$.

Let $S = \bigcup_{i=1}^r S_i$. Let V be a set such that $V \cap S = \varnothing$ and $|V| = k - n$. Let $\mathscr{T}$ consist of all sets $T = \{x_i\} \cup T' \cup V$, where $x_i \in X'$ and $T' \subseteq S \backslash X'$ satisfies

$|T'| = n-1$ and $T' \neq S_i \backslash \{x_i\}$. Then $S_i \not\subseteq T$ for all $S_i \in \mathscr{S}$ and $T \in \mathscr{T}$. Since $|S \backslash X'| = r(h-1)$, the family $\mathscr{T}$ consists of

$$t \geq r \binom{r(h-1)}{n-1} - r > cr^n$$

distinct sets of cardinality k.

Clearly, X' is a minimal system of representatives for $\mathscr{S}'$ and a system of representatives for $\mathscr{T}$, but

$$h^r \lambda^t < h^r \lambda^{cr^n} < 1 \leq N(\mathscr{S}', \mathscr{T}).$$

Choose $s > r$ so large that $t < \varepsilon s$. Let $S_{r+1}, \ldots, S_s$ be pairwise disjoint sets of cardinality h that are also disjoint from the sets in $\mathscr{S}'$ and $\mathscr{T}$. Let

$$\mathscr{S} = \mathscr{S}' \cup \{S_i\}_{i=r+1}^{s} = \{S_i\}_{i=1}^{s}.$$

There are h^{s-r} minimal systems of representatives for $\mathscr{S}$ that contain X', and so

$$h^s \lambda^t = h^{s-r} h^r \lambda^t < h^{s-r} \leq N(\mathscr{S}, \mathscr{T}).$$

The families $\mathscr{S}$ and $\mathscr{T}$ satisfy conditions (i)–(v) of the theorem.

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