PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 1, September 1988

AN APPLICATION OF BAUMSLAG-SOLITAR NONHOPFIAN GROUPS TO FINITARY AUTOMORPHISMS

FRANCESCO CASERTA

(Communicated by Bhama Srinivasan)

ABSTRACT. It is shown that finitary automorphisms, as distinct from those described as special, do not possess good hereditary properties on subgroups.

Hilton and Roitberg introduced in [**7**] two special classes of group automorphisms: $\varphi: G \to G$ is called a *finitary automorphism* if, for each $x \in G$, there exists a φ-invariant finitely generated subgroup $K_x \subseteq G$ such that $x \in K_x$ and $\varphi|K_x$ is an automorphism of K_x; φ is called a *special finitary automorphism* if, for each $x \in G$, the subgroup $H_x = \langle \varphi^n(x) \colon n \geq 0 \rangle$ is finitely generated and $\varphi|H_x$ is an automorphism of H_x. Actually, they were originally called *pseudo-identities* and *strong pseudo-identities*, respectively, but we prefer to use the more expressive terminology adopted in [**8**] (after a suggestion of G. Baumslag) and thereafter. These special automorphisms were introduced to extend the nice behaviour of the automorphisms of finitely generated nilpotent groups to automorphisms defined on a larger class of groups. For example, one would like to keep the property that, if

$$(*) \qquad \begin{array}{ccccc} G' & \rightarrowtail & G & \twoheadrightarrow & G'' \\ {\scriptstyle\varphi'}\big\downarrow & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\varphi''}\big\downarrow \\ G' & \rightarrowtail & G & \twoheadrightarrow & G'' \end{array}$$

is a short exact sequence of groups and endomorphisms, then φ is an automorphism if and only if φ' and φ'' are automorphisms.

Plainly, a special finitary automorphism is finitary, but there are finitary automorphisms which are not special. For example, the automorphism $\psi: F(x,y) \to F(x,y)$ of the free group on two generators x, y defined by $\psi(x) = x$ and $\psi(y) = x^{-1}yx$ is finitary, but not special because $y \notin \psi(H_y) = \langle x^{-n}yx^n \colon n \geq 1 \rangle$. U. Stammbach pointed out that the two notions coincide on locally noetherian groups. Hilton and Roitberg made an extensive study of the algebraic properties of finitary automorphisms in [**5**, **6**, **7**, **8**]. Just to mention a few results: finitary automorphisms of abelian groups are characterized by the property that for each $x \in G$ there exists a polynomial $p_x = \pm 1 + a_1 t + \cdots + a_{n-1}t^{n-1} \pm t^n \in \mathbb{Z}[t]$ such that $[p_x(\varphi)](x) = 0$; finitary automorphisms of abelian groups are closed under tensor and tor products; finitary automorphisms of nilpotent groups induce finitary automorphisms in homology. This opened up the way (through the Serre spectral sequence) to

Received by the editors December 4, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20E36; Secondary 20F99.
Key words and phrases. Group automorphism, finitary automorphism, nonhopfian group.

applications in homotopy theory, and in particular to self-maps of homologically nilpotent fibrations and of nilpotent spaces (see [**3**, **6**]). In fact, it was J. Cohen's earlier work [**4**] on pseudo-identities of abelian groups and self-maps of simply-connected fibrations that partially motivated Hilton and Roitberg to develop the theory for nonabelian groups. Recently the author in [**2**] has placed the work of Castellet, Hilton and Roitberg on finitary automorphisms in group theory and homotopy theory on an axiomatic foundation, relating it to the classical theory of Serre classes of groups.

Special finitary automorphisms behave nicely with respect to subgroups: the restriction of a special finitary automorphism φ to a φ-invariant subgroup is still a special finitary automorphism and if $(*)$ is a short exact sequence of groups and endomorphisms, then φ is a special finitary automorphism if and only if φ' and φ'' are. More generally, we have (see [**6**]) that an automorphism φ is special finitary if and only if $\varphi|H$ and $\varphi^{-1}|K$ are automorphisms for every φ-invariant subgroup H and φ^{-1}-invariant subgroup K. On the other hand, one has more freedom to decide whether a given automorphism $\varphi\colon G \to G$ is finitary by simply checking if there exists some finitely generated φ-invariant subgroup K_x containing x and such that $\varphi|K_x$ is bijective. We are not compelled to confine our attention to H_x.

However, finitary automorphisms in general are distinguished from special finitary automorphisms in that they do not possess good hereditary properties on subgroups. It is not difficult to find a finitary automorphism φ admitting a φ-invariant subgroup such that its restriction to that subgroup is not an automorphism. An example is provided again by ψ: if $H = \langle x^{-n}yx^n\colon\ n \geq 0\rangle$, then H is ψ-invariant and $\psi|H$ is not an automorphism because $y \notin \mathrm{Im}(\psi|H)$. Our basic observation to find similar examples, but with H normal, has been that in this case G/H cannot be hopfian, because otherwise the self-epimorphism induced by φ on G/H would be an automorphism, implying that $\varphi|H$ also would be an automorphism.

Let l and m be nonzero integers and let $G = \langle a, t\colon\ t^{-1}a^l t = a^m\rangle$. Baumslag and Solitar gave in [**1**] a necessary and sufficient condition for G to be nonhopfian, providing in this way the simplest possible examples of nonhopfian groups. If we take advantage of the fact that interchanging l and m above amounts merely to choosing an alternative presentation of G, we can say that G is nonhopfian if and only if $m \nmid l$ and there exists a prime integer p such that $p|l$ and $p \nmid m$.

THEOREM. *To each Baumslag-Solitar nonhopfian group G there exists an automorphism φ of the free group on three generators $F(x, y, z)$ and a normal φ-invariant subgroup H of $F(x, y, z)$ such that $\varphi|H$ is not an automorphism of H; thus φ is a finitary automorphism lacking the hereditary property on normal subgroups.*

PROOF. Assume $G = \langle a, t\colon\ t^{-1}a^l t = a^m\rangle$ is nonhopfian and let h and k be integers such that $hp^2 + km = 1$. Consider the diagram

$$\begin{array}{ccccc} H & \hookrightarrow & F(x,y,z) & \xrightarrow{\varepsilon} & G \\ {\scriptstyle \varphi|H}\downarrow & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle \psi}\downarrow \\ H & \hookrightarrow & F(x,y,z) & \xrightarrow{\varepsilon} & G \end{array}$$

defined by

$$\psi : \begin{cases} a \mapsto a^p, \\ t \mapsto t, \end{cases} \qquad \varphi : \begin{cases} x \mapsto y, \\ y \mapsto xz^{-1}y^{-kl/p}zy^{-p(h-1)}, \\ z \mapsto z, \end{cases} \qquad \varepsilon : \begin{cases} x \mapsto a, \\ y \mapsto a^p, \\ z \mapsto t, \end{cases}$$

and where $H = \ker \varepsilon$. The endomorphism ψ (introduced in [**1**]) is well-defined, the diagram is commutative (the condition $hp^2 + km = 1$ is essential here), φ is an automorphism (being the composite of Nielsen transformations) and ε is onto. As φ is an automorphism, ψ is onto. But ψ is not one-to-one, as shown in [**1**], and hence $\varphi|H$ is not onto.[1]

References

1. G. Baumslag and D. Solitar, *Some two-generator one-relator non-hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.
2. F. Caserta, *Serre classes of endomorphisms of nilpotent groups and self-maps of nilpotent spaces*, Ph.D. thesis, SUNY at Binghamton, 1988.
3. M. Castellet, P. Hilton and J. Roitberg, *On pseudo-identities*, II, Arch. Math. **42** (1984), 193–199.
4. J. Cohen, *A spectral sequence automorphism theorem; applications to fibre spaces and stable homotopy theory*, Topology **7** (1968), 173–177.
5. P. Hilton, *On special group automorphisms and their composition*, Canad. J. Math. **36** (1984), 591–600.
6. ———, *Special group automorphisms and special self-homotopy equivalences*, Aspects of Topology, London Math. Soc. Lecture Notes Series, vol. 93, Cambridge Univ. Press, 1985, pp. 281–292.
7. P. Hilton and J. Roitberg, *On pseudo-identities*, I, Arch. Math. **41** (1983), 204–214.
8. J. Roitberg, *Finitary automorphisms and integral homology*, Topological Topics, London Math. Soc. Lecture Notes Series, vol. 86, Cambridge Univ. Press, 1983, pp. 164–168.

Department of Mathematical Sciences, State University of New York, Binghamton, New York 13901

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

[1] An earlier example based on the group $\langle a, t \colon t^{-1}a^2t = a^3 \rangle$, the simplest in the Baumslag-Solitar family of nonhopfian groups, is due to Hilton and the author.