SIMULTANEOUS SYSTEMS OF REPRESENTATIVES
FOR FINITE FAMILIES OF FINITE SETS

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Abstract. Let \( h \geq 2 \) and \( k \geq 1 \). It is proved that if \( \mathcal{S} = \{S_i\}_{i=1}^s \) and \( \mathcal{T} = \{T_j\}_{j=1}^t \) are two families of nonempty, pairwise disjoint sets such that \( |S_i| \leq h \), \( |T_j| \leq k \) and \( S_i \not\subset T_j \) for all \( i \) and \( j \), then the number \( N(\mathcal{S}, \mathcal{T}) \) of the sets \( X \) such that \( X \) is a minimal system of representatives for \( \mathcal{S} \) and \( X \) is simultaneously a system of representatives for \( \mathcal{T} \) that satisfies \( N(\mathcal{S}, \mathcal{T}) < h^s(1 - (h - r)/h^{q+1})^t \), where \( k = q(h - 1) + r \) with \( 0 \leq r \leq h - 2 \). This was conjectured by M. B. Nathanson \[3\] in 1985.

1. Introduction. Let \( \mathcal{S} = \{S_i\} \) be a family of nonempty sets. The set \( X \) is a system of representatives for \( \mathcal{S} \) if \( X \cap S_i \neq \emptyset \) for every \( S_i \) in \( \mathcal{S} \). If \( X \) is a system of representatives for \( \mathcal{S} \), but no proper subset of \( X \) is a system of representatives for \( \mathcal{S} \), then \( X \) is called a minimal system of representatives for \( \mathcal{S} \).

Let \( \mathcal{S} = \{S_i\} \) and \( \mathcal{T} = \{T_j\} \) be two families of nonempty sets. Let \( N(\mathcal{S}, \mathcal{T}) \) denote the number of sets \( X \) such that \( X \) is a minimal system of representatives for \( \mathcal{S} \) and \( X \) is also a system of representatives for \( \mathcal{T} \).

The study of the number \( N(\mathcal{S}, \mathcal{T}) \) could be usefully applied to investigate asymptotic bases in additive number theory. In 1985, Nathanson \[3\] made two conjectures on this number, which can be stated as follows.

Conjecture 1. Let \( h \geq 2 \) and \( k \geq 1 \). There exists a real number \( \lambda = \lambda(h, k) \in (0, 1) \) with the following property: Let \( \mathcal{S} = \{S_i\}_{i=1}^s \) be a family of \( s \) nonempty, pairwise disjoint sets \( S_i \) with \( |S_i| \leq h \) for all \( i \). Let \( \mathcal{T} = \{T_j\}_{j=1}^t \) be a family of \( t \) nonempty, pairwise disjoint sets \( T_j \) with \( |T_j| \leq k \) for all \( j \). Suppose \( S_i \not\subset T_j \) for all \( i \) and \( j \). Then

\[
N(\mathcal{S}, \mathcal{T}) \leq h^s \lambda^t.
\]

Conjecture 2. Let \( h \geq 2 \) and \( k \geq 1 \). Let \( k = q(h - 1) + r \), where \( q = \lfloor k/(h-1) \rfloor \) and \( 0 \leq r \leq h - 2 \). Define

\[
\lambda^*(h, k) = 1 - (h - r)/h^{q+1}.
\]

Then \( \lambda^*(h, k) \) is the smallest value of \( \lambda \) for which inequality (1) is true for all families \( \mathcal{S} \) and \( \mathcal{T} \) that satisfy the conditions of Conjecture 1.

It has been proved that these two conjectures are true in many special cases. Early in 1979, Erdös and Nathanson \[1\] proved that the conjectures hold if \( h = k = 2 \) when they investigated asymptotic additive bases of order 2 in additive number theory. Jia \[2\] proved in 1986 that the conjectures are true in the case that...
In his 1985 paper [3], Nathanson proved the conjectures in some special cases. And he proved that if \( \lambda \in (0,1) \) satisfies \( N(\mathcal{S}, \mathcal{T}) \leq h^s \lambda^t \) for all \( \mathcal{S} \) and \( \mathcal{T} \) that satisfy the conditions of Conjecture 1, then \( \lambda \geq \lambda^*(h,k) \). In the present paper, we prove that Conjectures 1 and 2 are true for any \( h \geq 2 \) and \( k \geq 1 \).

2. Main result and a lemma. The main result of this paper is

**Theorem.** Let \( h \geq 2 \) and \( k \geq 1 \). Let \( k = q(h-1) + r \), where \( q = \lfloor k/(h-1) \rfloor \) and \( 0 \leq r \leq h-2 \). Then

\[
N(\mathcal{S}, \mathcal{T}) \leq h^s(1 - (h - r)/hq + 1)^t
\]

holds for any finite families \( \mathcal{S} \) and \( \mathcal{T} \) that satisfy the conditions of Conjecture 1.

In particular, we have

\[
N(\mathcal{S}, \mathcal{T}) \leq h^s((h^2 - h + 1)/h^2)^t
\]

if \( h = k \geq 2 \), which is a result by Jia [2], and

\[
N(\mathcal{S}, \mathcal{T}) \leq h^s(k/h)^t
\]

if \( h > k \geq 1 \).

The following lemma will be used in the proof of the theorem.

**Lemma.** Let \( h \geq 2 \) and \( m \geq 1 \) with \( m \leq L < mh \). If

\[
L - m = u(h - 1) - r,
\]

where \( u \) is an integer and \( 0 \leq r \leq h - 2 \), then

\[
x_1 \cdots x_m \geq h^{u-1}(h - r)
\]

holds for any integers \( 1 \leq x_i \leq h \) \( (i = 1, 2, \ldots, m) \) with \( \sum_{i=1}^{m} x_i = L \).

**Proof.** Let \( f(x_1, \ldots, x_m) = x_1 \cdots x_m \). It is well known that \( f \) has no minimal point inside the inner \( \mathcal{D} \) of the domain \( \overline{\mathcal{D}} : 1 \leq x_i \leq h \) \( (i = 1, \ldots, m) \) with the restriction \( x_1 + \cdots + x_m = L \). Hence the minimal point of \( f \) must be on the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \).

Since \( L < mh \), it follows from the definition of \( u \) that \( u \leq m \). First we assume \( u = m \), then \( L = mh - r \). We prove

\[
f(x_1, \ldots, x_m) \geq h^{m-1}(h - r)
\]

by induction on \( m \). It is clear that (5) is true if \( m = 1 \). Now assume that (5) holds for any \( m' < m \). Let \( (x_1, \ldots, x_m) \) be a minimal point of \( f \) on \( \partial \mathcal{D} \), where \( x_1 \leq x_2 \leq \cdots \leq x_m \). Since \( x_1 = L - (x_2 + \cdots + x_m) \geq L - (m - 1)h = h - r \geq 2 \), it follows from \( (x_1, \ldots, x_m) \in \partial \mathcal{D} \) that \( x_m = h \). Therefore \( x_1 + \cdots + x_{m-1} = L - r = (m - 1)h - r \), thus

\[
f(x_1, \ldots, x_m) = x_1 \cdots x_m = hx_1 \cdots x_{m-1} \\
\geq h(h^{m-2}(h - r)) = h^{m-1}(h - r),
\]

which proves (5).
Now assume $u < m$. If $(x_1, \ldots, x_m) \in \partial \mathbb{D}$ is such that $x_1 \leq \cdots \leq x_m$ and $f(x_1, \ldots, x_m)$ is minimal, then $x_1 = 1$. Otherwise, we suppose $2 \leq x_1 \leq \cdots \leq x_u < x_{u+1} = \cdots = x_m = h$. Then $(x_1 - 1, x_2, \ldots, x_{u-1}, x_u + 1, x_{u+1}, \ldots, x_m) \in \partial \mathbb{D}$, and

$$f(x_1 - 1, x_2, \ldots, x_{u-1}, x_u + 1, x_{u+1}, \ldots, x_m) = (x_1 - 1)x_2 \cdots x_{u-1}(x_u + 1)^{h^{m-u}} = x_1x_2 \cdots x_u h^{m-u} - x_2 \cdots x_{u-1}(x_u + 1 - x_1) < f(x_1, x_2, \ldots, x_m),$$

which contradicts the minimality of $f(x_1, \ldots, x_m)$. Therefore $x_2 + \cdots + x_m = L - 1 = u(h - 1) - r + (m - 1)$, thus $x_1 \cdots x_m = x_2 \cdots h^{u-1}(h - r)$. This shows that we can assume that $u = m$. Hence the proof of the lemma is complete.

3. The proof of the theorem. Let $\mathcal{S} = \{S_i\}_{i=1}^s$ and $\mathcal{T} = \{T_j\}_{j=1}^t$ be two finite families of finite sets that satisfy the conditions of Conjecture 1. Let $S_{s+1}$ be a set of $h$ elements such that $S_{s+1}$ does not intersect any $S_i$ in $\mathcal{S}$. Taking $\mathcal{S}' = \mathcal{S} \cup \{S_{s+1}\}$, we have $N(\mathcal{S}', \mathcal{T}) \geq hN(\mathcal{S}, \mathcal{T})$. This allows us to assume that the integer $s$ is sufficiently large. Therefore we may assume without loss of generality that

$$|S_i| = h \quad \text{for } i = 1, 2, \ldots, s;$$
$$|T_j| = k \quad \text{for } j = 1, 2, \ldots, t;$$

and

$$S = \bigcup_{i=1}^s S_i \supseteq T = \bigcup_{j=1}^t T_j.$$

We will prove the theorem by induction on $t$ for fixed $s > 2kt$. If $t = 0$ then $N(\mathcal{S}, \mathcal{T}) = h^s$. Let $t \geq 1$ and assume that (2) holds for any $0 \leq t' < t$ and any $s$.

We consider $T_t$. Let $\{S_1, \ldots, S_m\}$ be the set of those $S_i$ that intersect $T_t$. Denote

$$|S_i \cap T_t| = n_i \quad \text{for } i = 1, \ldots, m,$$

then $n_1 + \cdots + n_m = k$, and $S_i \not\subseteq T_t$ implies that $1 \leq n_i \leq h - 1$ for $i = 1, \ldots, m$. Let $S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\}$, where $a_{i1}, \ldots, a_{ih} \in T_t$ for $i = 1, \ldots, m$. Since $s > 2kt$, there exist $m$ $S_i$ in $\mathcal{S}$, say $S_{m+1}, \ldots, S_{2m}$, such that $S_i \cap T = \emptyset$ for $i = m + 1, \ldots, 2m$. Let $S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\}$ for $i = m + 1, \ldots, 2m$.

We construct

$$S'_i = \{a_{i1}, \ldots, a_{in}, a_{m+i,n_i+1}, \ldots, a_{m+i,h}\},$$
$$S'_{m+i} = \{a_{m+i,1}, \ldots, a_{m+i,n_i}, a_{i,n_i+1}, \ldots, a_{ih}\}$$

for $i = 1, \ldots, m$. Let

$$\mathcal{S}' = (\mathcal{S} \setminus \{S_1, \ldots, S_{2m}\}) \cup \{S'_1, \ldots, S'_{2m}\}.$$

Then $\mathcal{S}'$ and $\mathcal{T}$ satisfy the conditions of Conjecture 1, and the corresponding integers $s$ and $t$ do not change.

Let $X$ be a simultaneous system of representatives counted in $N(\mathcal{S}, \mathcal{T})$. Denote

$$X \cap S_i = \{x_i\} \quad \text{for } i = 1, \ldots, m;$$
$$X_1 = X \bigcup_{i=1}^{2m} S_i.$$
Then it follows from $S_i \cap T = \emptyset$ for $i = m+1, \ldots, 2m$ that exactly $h^m$ simultaneous systems $X$ of representatives counted in $N(\mathcal{S}, \mathcal{R})$ contain $\{x_1, \ldots, x_m\} \cup X_1$.

Suppose $x_j \in S_{ij}^t$ for $j = 1, \ldots, m$. Then $i_j = j$ or $j + m$, and $S_{ij}^t + m \cap (\{x_1, \ldots, x_m\} \cup X_1) = \emptyset$ for $j = 1, \ldots, m$, where the subscripts of $S_{ij}^t$'s are regarded as elements of the group $\mathbb{Z}/(2m)$. Therefore for any $x_j + m \in S_{ij}^t + m$ for $j = 1, \ldots, m$, taking $X' = \{x_1, \ldots, x_{2m}\} \cup X_1$, we see that $X'$ is a minimal system of representatives for $\mathcal{S}'$ that contains a system of representatives for $\mathcal{R}$, and $X'$ contains $\{x_1, \ldots, x_m\} \cup X_1$. Since there are $h^m$ different simultaneous systems $X$ of representatives counted in $N(\mathcal{S}'^t, \mathcal{R})$. Therefore $N(\mathcal{S}'^t, \mathcal{R}) \leq N(\mathcal{S}^t, \mathcal{R})$. Hence we may assume that $S_i \cap T_t = S_i \cap T$ for $i = 1, 2, \ldots, m$.

Let $\mathcal{S}' = \{S_i\}_{i=m+1}^t$ and $\mathcal{R}' = \{T_j\}_{j=1}^t$. Clearly $\mathcal{S}'$ and $\mathcal{R}'$ satisfy the conditions of Conjecture 1 (for $s = m$ and $t' = 1$). For any $X_1$ counted in $N(\mathcal{S}'^t, \mathcal{R}')$, there are $h^m - (h - n_1) \cdots (h - n_m)$ different $X$ counted in $N(\mathcal{S}^t, \mathcal{R})$ containing $X_1$. Since

$$L = \sum_{i=1}^m (h - n_i) = mh - \sum_{i=1}^m n_i = mh - k,$$

then $L - m = m(h - 1) - k = m(h - 1) - q(h - 1) - r = (m - q)(h - 1) - r$. Hence by the lemma, we have $(h - n_1) \cdots (h - n_m) \geq h^{m-q-1}(h - r)$. Therefore

$$N(\mathcal{S}, \mathcal{R}) \leq \left( h^m - (h - n_1) \cdots (h - n_m) \right) N(\mathcal{S}'^t, \mathcal{R}') \leq \left( h^m - h^{m-q-1}(h - r) \right) h^{s-m} (1 - (h - r)/h^{q+1})^{t-1} = h^s (1 - (h - r)/h^{q+1})^t.$$

This completes the proof of the theorem.

Nathanson [3] has given an example of two families $\mathcal{S}$ and $\mathcal{R}$ of finite sets that satisfy the conditions of Conjecture 1, for which,

$$N(\mathcal{S}, \mathcal{R}) = h^s (1 - (h - r)/h^{q+1})^t.$$

Therefore the upper bound of $N(\mathcal{S}, \mathcal{R})$ in the theorem above is the best possible result.

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REFERENCES


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