SIMULTANEOUS SYSTEMS OF REPRESENTATIVES
FOR FINITE FAMILIES OF FINITE SETS

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Abstract. Let \( h > 2 \) and \( k > 1 \). It is proved that if \( \mathcal{S} = \{S_i\}_{i=1}^s \) and \( \mathcal{T} = \{T_j\}_{j=1}^t \) are two families of nonempty, pairwise disjoint sets such that \( |S_i| \leq h, |T_j| \leq k \) and \( S_i \not\subseteq T_j \) for all \( i \) and \( j \), then the number \( N(\mathcal{S},\mathcal{T}) \) of the sets \( X \) such that \( X \) is a minimal system of representatives for \( \mathcal{S} \) and \( X \) is simultaneously a system of representatives for \( \mathcal{T} \) that satisfies \( N(\mathcal{S},\mathcal{T}) < h^s(1 - (h - r)/h^{q+1})^t \), where \( k = q(h - 1) + r \) with \( 0 < r \leq h - 2 \). This was conjectured by M. B. Nathanson [3] in 1985.

1. Introduction. Let \( \mathcal{S} = \{S_i\} \) be a family of nonempty sets. The set \( X \) is a system of representatives for \( \mathcal{S} \) if \( X \cap S_i \neq \emptyset \) for every \( S_i \) in \( \mathcal{S} \). If \( X \) is a system of representatives for \( \mathcal{S} \), but no proper subset of \( X \) is a system of representatives for \( \mathcal{S} \), then \( X \) is called a minimal system of representatives for \( \mathcal{S} \).

Let \( \mathcal{S} = \{S_i\} \) and \( \mathcal{T} = \{T_j\} \) be two families of nonempty sets. Let \( N(\mathcal{S},\mathcal{T}) \) denote the number of sets \( X \) such that \( X \) is a minimal system of representatives for \( \mathcal{S} \) and \( X \) is also a system of representatives for \( \mathcal{T} \).

The study of the number \( N(\mathcal{S},\mathcal{T}) \) could be usefully applied to investigate asymptotic bases in additive number theory. In 1985, Nathanson [3] made two conjectures on this number, which can be stated as follows.

**Conjecture 1.** Let \( h \geq 2 \) and \( k \geq 1 \). There exists a real number \( \lambda = \lambda(h,k) \in (0,1) \) with the following property: Let \( \mathcal{S} = \{S_i\}_{i=1}^s \) be a family of \( s \) nonempty, pairwise disjoint sets \( S_i \) with \( |S_i| \leq h \) for all \( i \). Let \( \mathcal{T} = \{T_j\}_{j=1}^t \) be a family of \( t \) nonempty, pairwise disjoint sets \( T_j \) with \( |T_j| \leq k \) for all \( j \). Suppose \( S_i \not\subseteq T_j \) for all \( i \) and \( j \). Then

\[
N(\mathcal{S},\mathcal{T}) \leq h^s \lambda^t.
\]

**Conjecture 2.** Let \( h \geq 2 \) and \( k \geq 1 \). Let \( k = q(h - 1) + r \), where \( q = \lfloor k/(h-1) \rfloor \) and \( 0 \leq r \leq h - 2 \). Define

\[
\lambda^*(h,k) = 1 - (h - r)/h^{q+1}.
\]

Then \( \lambda^*(h,k) \) is the smallest value of \( \lambda \) for which inequality (1) is true for all families \( \mathcal{S} \) and \( \mathcal{T} \) that satisfy the conditions of Conjecture 1.

It has been proved that these two conjectures are true in many special cases. Early in 1979, Erdős and Nathanson [1] proved that the conjectures hold if \( h = k = 2 \) when they investigated asymptotic additive bases of order 2 in additive number theory. Jia [2] proved in 1986 that the conjectures are true in the case that
h = k ≥ 2. In his 1985 paper [3], Nathanson proved the conjectures in some special cases. And he proved that if λ ∈ (0, 1) satisfies N(𝒮, ℱ) ≤ h^nλ^t for all 𝒮 and ℱ that satisfy the conditions of Conjecture 1, then λ ≥ λ*(h, k). In the present paper, we prove that Conjectures 1 and 2 are true for any h ≥ 2 and k ≥ 1.

2. Main result and a lemma. The main result of this paper is

**Theorem.** Let h ≥ 2 and k ≥ 1. Let k = q(h - 1) + r, where q = ⌊k/(h - 1)⌋ and 0 ≤ r ≤ h - 2. Then

(2) \[ N(𝒮, ℱ) ≤ h^n(1 - r/h(q + 1)) \]

holds for any finite families 𝒮 and ℱ that satisfy the conditions of Conjecture 1.

In particular, we have

\[ N(𝒮, ℱ) ≤ h^n((h^2 - h + 1)/h^2) \]

if h = k ≥ 2, which is a result by Jia [2], and

\[ N(𝒮, ℱ) ≤ h^n(k/h) \]

if h > k ≥ 1.

The following lemma will be used in the proof of the theorem.

**Lemma.** Let h ≥ 2 and m ≥ 1 with m ≤ L < mh. If

(3) \[ L - m = u(h - 1) - r, \]

where u is an integer and 0 ≤ r ≤ h - 2, then

(4) \[ x_1 \cdots x_m ≥ h^{u-1}(h - r) \]

holds for any integers 1 ≤ x_i ≤ h (i = 1, 2, ..., m) with \( \sum_{i=1}^{m} x_i = L \).

**Proof.** Let \( f(x_1, ..., x_m) = x_1 \cdots x_m \). It is well known that \( f \) has no minimal point inside the inner \( \mathcal{D} \) of the domain \( \overline{\mathcal{D}}: 1 ≤ x_i ≤ h \) (i = 1, ..., m) with the restriction \( x_1 + \cdots + x_m = L \). Hence the minimal point of \( f \) must be on the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \).

Since \( L < mh \), it follows from the definition of \( u \) that \( u ≤ m \). First we assume \( u = m \), then \( L = mh - r \). We prove

(5) \[ f(x_1, ..., x_m) ≥ h^{m-1}(h - r) \]

by induction on \( m \). It is clear that (5) is true if \( m = 1 \). Now assume that (5) holds for any \( m' < m \). Let \( (x_1, ..., x_m) \) be a minimal point of \( f \) on \( \partial \mathcal{D} \), where \( x_1 ≤ x_2 ≤ \cdots ≤ x_m \). Since \( x_1 = L - (x_2 + \cdots + x_m) ≥ L - (m - 1)h = h - r ≥ 2 \), it follows from \( (x_1, ..., x_m) \in \partial \mathcal{D} \) that \( x_m = h \). Therefore \( x_1 + \cdots + x_{m-1} = L - r = (m - 1)h - r \), thus

\[ f(x_1, ..., x_m) = x_1 \cdots x_m = hx_1 \cdots x_{m-1} ≥ h(h^{m-2}(h - r)) = h^{m-1}(h - r), \]

which proves (5).
Now assume \( u < m \). If \((x_1, \ldots, x_m) \in \partial \mathcal{D}\) is such that \( x_1 \leq \cdots \leq x_m \) and \( f(x_1, \ldots, x_m) \) is minimal, then \( x_1 = 1 \). Otherwise, we suppose \( 2 \leq x_1 \leq \cdots \leq x_u < x_{u+1} = \cdots = x_m = h \). Then \((x_1-1, x_2, \ldots, x_{u-1}, x_u+1, x_{u+1}, \ldots, x_m) \in \partial \mathcal{D}\), and
\[
f(x_1-1, x_2, \ldots, x_{u-1}, x_u+1, x_{u+1}, \ldots, x_m) = (x_1-1)x_2 \cdots x_{u-1}(x_u+1)h^{m-u}
= x_1x_2 \cdots x_u h^{m-u} - x_2 \cdots x_{u-1}h^{m-u}(x_u+1-x_1) < f(x_1, x_2, \ldots, x_m),
\]
which contradicts the minimality of \( f(x_1, \ldots, x_m) \). Therefore \( x_2 + \cdots + x_m = L-1 = u(h-1) - r + (m-1) \), thus \( x_1 \cdots x_m = x_2 \cdots h^{u-1}(h-r) \). This shows that we can assume that \( u = m \). Hence the proof of the lemma is complete.

3. The proof of the theorem. Let \( \mathcal{S} = \{S_i\}_{i=1}^s \) and \( \mathcal{T} = \{T_j\}_{j=1}^t \) be two finite families of finite sets that satisfy the conditions of Conjecture 1. Let \( S_{s+1} \) be a set of \( h \) elements such that \( S_{s+1} \) does not intersect any \( S_i \) in \( \mathcal{S} \). Taking \( \mathcal{S}' = \mathcal{S} \cup \{S_{s+1}\} \), we have \( N(\mathcal{S}', \mathcal{T}) \geq hN(\mathcal{S}, \mathcal{T}) \). This allows us to assume that the integer \( s \) is sufficiently large. Therefore we may assume without loss of generality that
\[
|S_i| = h \quad \text{for } i = 1, 2, \ldots, s;
|T_j| = k \quad \text{for } j = 1, 2, \ldots, t;
\]
and
\[
S = \bigcup_{i=1}^s S_i \supseteq T = \bigcup_{j=1}^t T_j.
\]

We will prove the theorem by induction on \( t \) for fixed \( s > 2kt \). If \( t = 0 \) then \( N(\mathcal{S}, \mathcal{T}) = h^s \). Let \( t \geq 1 \) and assume that (2) holds for any \( 0 \leq t' < t \) and any \( s \).

We consider \( T_t \). Let \( \{S_1, \ldots, S_m\} \) be the set of those \( S_i \) that intersect \( T_t \). Denote
\[
|S_i \cap T_t| = n_i \quad \text{for } i = 1, \ldots, m,
\]
then \( n_1 + \cdots + n_m = k \), and \( S_i \not\subset T_t \) implies that \( 1 \leq n_i \leq h-1 \) for \( i = 1, \ldots, m \). Let \( S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\} \), where \( a_{i1}, \ldots, a_{ih} \in T_t \) for \( i = 1, \ldots, m \).

Since \( s > 2kt \), there exist \( m \) \( S_i \) in \( \mathcal{S} \), say \( S_{m+1}, \ldots, S_{2m} \), such that \( S_i \cap T_t = \emptyset \) for \( i = m+1, \ldots, 2m \). Let \( S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\} \) for \( i = m+1, \ldots, 2m \).

We construct
\[
S'_i = \{a_{i1}, \ldots, a_{ih}, a_{m+i,n_i+1}, \ldots, a_{m+i,h}\},
S'_{m+i} = \{a_{m+i,1}, \ldots, a_{m+i,n_i}, a_{i,n_i+1}, \ldots, a_{ih}\}
\]
for \( i = 1, \ldots, m \). Let
\[
\mathcal{S}' = (\mathcal{S} \setminus \{S_1, \ldots, S_{2m}\}) \cup \{S'_1, \ldots, S'_{2m}\}.
\]
Then \( \mathcal{S}' \) and \( \mathcal{T} \) satisfy the conditions of Conjecture 1, and the corresponding integers \( s \) and \( t \) do not change.

Let \( X \) be a simultaneous system of representatives counted in \( N(\mathcal{S}, \mathcal{T}) \). Denote
\[
X \cap S_i = \{x_i\} \quad \text{for } i = 1, \ldots, m;
X_1 = X \bigcup_{i=1}^{2m} S_i.
\]
Then it follows from $S_i \cap T = \emptyset$ for $i = m+1, \ldots, 2m$ that exactly $h^m$ simultaneous systems $X$ of representatives counted in $N(\mathcal{S}, \mathcal{F})$ contain $\{x_1, \ldots, x_m\} \cup X_1$.

Suppose $x_j \in S'_j$ for $j = 1, \ldots, m$. Then $i_j = j$ or $j + m$, and $S'_{i_j + m} \cap (\{x_1, \ldots, x_m\} \cup X_1) = \emptyset$ for $j = 1, \ldots, m$, where the subscripts of $S'_j$'s are regarded as elements of the group $\mathbb{Z}/(2m)$. Therefore for any $x_{i_j + m} \in S'_{i_j + m}$ for $j = 1, \ldots, m$, taking $X' = \{x_1, \ldots, x_{2m}\} \cup X_1$, we see that $X'$ is a minimal system of representatives for $\mathcal{S}'$ that contains a system of representatives for $\mathcal{F}$, and $X'$ contains $\{x_1, \ldots, x_m\} \cup X_1$. Since there are $h^m$ different simultaneous systems $X$ of representatives counted in $N(\mathcal{S}'^m, \mathcal{F})$. Therefore $N(\mathcal{S}'^m, \mathcal{F}) \leq N(\mathcal{S}, \mathcal{F})$. Hence we may assume that $S_i \cap T_i = S_i \cap T$ for $i = 1, 2, \ldots, m$.

Let $\mathcal{S}' = \{S_i\}_{i=m+1}^s$ and $\mathcal{F}' = \{T_j\}_{j=1}^t$. Clearly $\mathcal{S}'$ and $\mathcal{F}'$ satisfy the conditions of Conjecture 1 (for $s - m$ and $t - 1$). For any $X_1$ counted in $N(\mathcal{S}'^m, \mathcal{F}')$, there are $h^m - (h - n_1)\cdots(h - n_m)$ different $X$ counted in $N(\mathcal{S}, \mathcal{F})$ containing $X_1$. Since

$$L = \sum_{i=1}^m (h - n_i) = mh - \sum_{i=1}^m n_i = mh - k,$$

then $L - m = m(h - 1) - k = m(h - 1) - q(h - 1) - r = (m - q)(h - 1) - r$. Hence by the lemma, we have $(h - n_1)\cdots(h - n_m) \geq h^{m-q-1}(h - r)$. Therefore

$$N(\mathcal{S}, \mathcal{F}) \leq (h^m - (h - n_1)\cdots(h - n_m))N(\mathcal{S}'^m, \mathcal{F}')$$

$$\leq (h^m - h^{m-q-1}(h - r))h^{s-m}(1 - (h - r)/h^{q+1})^{t-1}$$

$$= h^s(1 - (h - r)/h^{q+1})^t.$$

This completes the proof of the theorem.

Nathanson [3] has given an example of two families $\mathcal{S}$ and $\mathcal{F}$ of finite sets that satisfy the conditions of Conjecture 1, for which,

$$N(\mathcal{S}, \mathcal{F}) = h^s(1 - (h - r)/h^{q+1})^t.$$

Therefore the upper bound of $N(\mathcal{S}, \mathcal{F})$ in the theorem above is the best possible result.

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