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A CHARACTERIZATION OF ARTINIAN RINGS WHOSE ENDOMORPHISM RINGS HAVE FINITE GLOBAL DIMENSION

DAN ZACHARIA

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ABSTRACT. We prove that if Λ is a left artinian ring having the property that for every idempotent e, the ring $e \wedge e$ has finite global dimension then Λ is a quotient of an hereditary artinian ring.

1. Introduction. 1.1 Let Λ be a left artinian ring of finite global dimension, and let e be an idempotent in Λ. It is not true in general that for every primitive idempotent e, $\operatorname{gl\,dim} e\Lambda e$ remains finite; E. Green has provided a counterexample in [**1**].

The purpose of this paper is to prove the following result:

THEOREM. *Let Λ be an artinian ring having the property that* $\operatorname{gl\,dim} \operatorname{End}_\Lambda(P) < \infty$ *for every finitely generated projective Λ module P. Then Λ is a quotient of an hereditary artinian ring.*

1.2 Let us note that Λ is a quotient of an hereditary artinian ring if and only if there is no closed "oriented" chain of nonzero homomorphisms, nonisomorphisms

$$P_{i_1} \to P_{i_2} \to \cdots \to P_{i_s} \to P_{i_1}$$

through indecomposable projective Λ modules. An equivalent statement is that the global dimension of $\Lambda/\underline{r}^2$ is finite where $\underline{r}$ is the Jacobson radical of Λ. These facts have been proven by Jans and Nakayama [**2**], and it is very easy to see that if Λ is a quotient of an hereditary artinian ring then $\operatorname{gl\,dim}\operatorname{End}_\Lambda(P) < \infty$ for every projective Λ module P.

2. The main result. Let Λ be an artinian ring with the property that

$$\operatorname{gl\,dim}\operatorname{End}_\Lambda(P) < \infty$$

for every projective Λ-module P. Let $P_1, P_2, \ldots, P_n$ be a complete set of nonisomorphic indecomposable projective Λ-modules, and let

$$Q_i = \coprod_{j \neq i} P_j \quad \text{and} \quad \Lambda_i = \operatorname{End}_\Lambda(Q_i)^{\mathrm{op}} \quad \text{for } i = 1, 2, \ldots, n.$$

LEMMA. *If there exists a simple projective Λ_i module, then there is an index $k \neq i$ such that either P_k is simple or $\underline{r}P_k$ is isomorphic to a direct sum of copies of the simple module $P_i/\underline{r}P_i$.*

PROOF. The indecomposable projective Λ_i modules are (up to isomorphism) the modules $\operatorname{Hom}_\Lambda(Q_i, P_j)$, $j = 1, \ldots, n$, and $j \neq i$: Suppose that $\operatorname{Hom}_\Lambda(Q_i, P_k)$ is simple. Then we have $\operatorname{Hom}_\Lambda(Q_i, \underline{r}P_k) = 0$. Now our claim follows since $\operatorname{Hom}_\Lambda(P_i, P_i)$ is semisimple artinian by assumptions. □

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We can now prove our main result.

PROOF OF THEOREM. We proceed by induction on n. Clearly we may assume $\underline{r}^2 \neq 0$ and $n > 1$.

By induction hypothesis $\Lambda_1, \ldots, \Lambda_n$ are quotients of hereditary artinian rings, thus by [**2**] there exists a simple projective Λ_i module for each i. Since $r^2 \neq 0$, the lemma yields a simple projective Λ-module (otherwise all P_i would have Loewy length 2). Say P_1 is simple projective. Then, since there are no closed oriented chains of nonzero homomorphisms, nonisomorphisms among $P_2, \ldots, P_n$ by induction, no such chains arise among $P_1, \ldots, P_n$. This completes the proof. $\square$

REFERENCES

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DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244