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THE ISOMORPHISM QUESTION FOR MODULAR GROUP ALGEBRAS OF METACYCLIC p-GROUPS

CZESŁAW BAGIŃSKI

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ABSTRACT. Let $F[G]$ be a group algebra of a finite p-group G over the field $F = GF(p)$. If $G \simeq H$, then clearly $F[G] \simeq F[H]$. However, it is not known whether the converse is true. The answer for metacyclic p-groups, $p > 3$, is given.

Let p be a prime, G—a finite nonabelian p-group and $F = GF(p)$—the field with p elements. In this note we study the following isomorphism problem: Whether the isomorphism of group algebras $F[G]$ and $F[H]$ implies the isomorphism of groups G and H. In [**4**] it was given the positive answer to this question for all p-groups of order at most p^4. Here we prove the following

THEOREM. *If G is a metacyclic p-group, $p > 3$, and $F[G] \simeq F[H]$, then $G \simeq H$.*

Our result depends on some lemmas. We use standard notation, see [**3**, **5**].

LEMMA 1. *Let N be a normal subgroup of a finite p-group G. If $I = \omega(F[N])F[G]$, then $I^n = \omega(F[N])^n F[G]$. Moreover, if η_i, $i = 1, \dots, t$, form a basis of the space $\omega(F[N])^n$ and $\{1 = g_1, \dots, g_s\}$ is a right transversal for N in G, then $\eta_i g_j$, $i = 1, \dots, t$, $j = 1, \dots, s$, form a basis of the space I^n.*

The proof of the first part is by an easy induction on n. The second part is obvious.

For any finite p-group G let $\{\mathscr{M}_i(G)\}$ be the Brauer-Jennings-Zassenhaus $\mathscr{M}$-series of a p-group G defined by $\mathscr{M}_1(G) = G$ and for $n \geq 2$

$$\mathscr{M}_n(G) = (\mathscr{M}_{n-1}(G), G)\mathscr{M}_i(G)^{(p)},$$

where i is the smallest integer satisfying $ip \geq n$.

LEMMA 2. *If G' is the commutator subgroup of G, then the factor groups $\mathscr{M}_i(G')/\mathscr{M}_{i+1}(G')$ for all i are determined by $F[G]$.*

PROOF. By proof of [**5**, 14.2.7i] the factor groups $\mathscr{M}_i(G')/\mathscr{M}_{i+1}(G')$ are determined up to isomorphism by

$$\begin{aligned} f_j &= \dim_F \omega(F[G'])^j/\omega(F[G'])^{j+1} \\ &= \frac{1}{|G\colon G'|} \dim_F \omega(F[G'])^j F[G]/\omega(F[G'])^{j+1} F[G]. \end{aligned}$$

Since $\omega(F[G'])F[G]$ as an ideal generated by the subspace $[F[G], F[G]]$, is determined by $F[G]$ and $|G\colon G'| = \dim_F F[G]/\omega(F[G'])F[G]$ the result follows from Lemma 1.

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By the proof of [**5**, 14.2.7ii] we obtain then

COROLLARY 1. *If G' is cyclic, then the isomorphism class of G' is determined by $F[G]$. In particular, if $F[G] \simeq F[H]$ and G' is cyclic, then H' is cyclic too.*

Remark now that if G is a metacyclic p-group, then by the inclusion $(\mathscr{M}_n(G), G) \subset \mathscr{M}_n(G)^{(p)}$ we have $\mathscr{M}_2(G) = G^{(p)} = \mathscr{M}_3(G) = \cdots = \mathscr{M}_p(G)$, $\mathscr{M}_{p+1}(G) = \cdots = \mathscr{M}_{p^2}(G) = G^{(p^2)}$ and so on. Thus we have

COROLLARY 2. *If G is a metacyclic p-group then the exponent of G is determined by $F[G]$.*

LEMMA 3. *If G is a finite p-group with G' cyclic, then for every integer $n \geq 1$*

$$\omega(F[G'^{(p^n)}])F[G] = (\omega(F[G'])F[G])^{p^n}.$$

PROOF. Let $G' = \langle g \rangle$. Elements $g-1, (g-1)^2, \ldots, (g-1)^{o(g)-1}$ form a basis of $\omega(F[G'])$ [**5**, 3.3.3]. So $\omega(F[G'])^{p^n} \subset \omega(F[G'^{(p^n)}])F[G] \subset \omega(F[G'])^{p^n}F[G]$ and then by Lemma 1 $\omega(F[G'^{(p^n)}])F[G] = (\omega(F[G'])F[G])^{p^n}$.

Let now $\Omega_1(G) = \langle x \in G | x^p = 1 \rangle$. We shall say that G is a p_1-group iff $x^p = y^p$ implies $(xy^{-1})^p = 1$ for all $x, y \in G$. Let $\omega_1(F[G])$ be the ideal of $F[G]$ generated by all elements $a \in F[G]$ satisfying $a^p = 0$.

LEMMA 4 [**1**, LEMMA 3]. *If G is a p_1-group and for all $g \in G$ $g^p \in Z(G)$, then $\omega_1(F[G]) = \omega(F[H])F[G]$, where $H = \Omega_1(G)$.*

PROOF OF THE THEOREM. Let $F[G] \simeq F[H]$, where G is metacyclic p-group and $p > 3$. First we show that H is metacyclic. Let $\overline{G}$ and $\overline{H}$ be the factor groups $G/G'^{(p)}$, $H/H'^{(p)}$ respectively. Since by Lemma 3 $(\omega(F[G'])F[G])^p = \omega(F[G'^{(p)}])F[G]$ we have

$$F[\overline{G}] \simeq F[G]/(\omega(F[G'])F[G])^p \simeq F[H]/(\omega(F[H'])F[H])^p \simeq F[\overline{H}].$$

The group $\overline{G}$ satisfies the assumption of Lemma 4, so

$$\omega_1(F[\overline{G}]) = \omega(F[\Omega_1(\overline{G})])F[\overline{G}].$$

By the obvious inclusion $\omega(F[\Omega_1(\overline{H})])F[\overline{H}] \subset \omega_1(F[\overline{H}])$ we have then

$$\begin{aligned} |\overline{H}/\Omega_1(\overline{H})| &= \dim_F F[\overline{H}/\Omega_1(\overline{H})] = \dim_F F[\overline{H}] - \dim_F \omega(F[\Omega_1(\overline{H})])F[\overline{H}] \\ &\geq \dim_F F[\overline{H}] - \dim_F \omega_1(F[\overline{H}]) = \dim_F F[\overline{G}] - \dim_F \omega_1(F[\overline{G}]) \\ &= \dim_F F[\overline{G}/\Omega_1(\overline{G})] = |\overline{G}/\Omega_1(\overline{G})| \end{aligned}$$

which implies $|\Omega_1(\overline{H})| \leq |\Omega_1(\overline{G})| = p^2$. Hence, since $\overline{H}$ is nonabelian and $p > 2$, $|\Omega_1(\overline{H})| = p^2$. Now, by [**3**, III.11.6] $\overline{H}$ is metacyclic. But we have assumed that $p > 3$ and from Corollary 1 H' is cyclic, that is $H'^{(p)} = \Phi(H')\gamma_3(H)$. So by [**3**, III.11.3] H is metacyclic too. The fact that $\overline{H}$ is metacyclic and more, that $\overline{G} \simeq \overline{H}$, one can obtain also from [**6**, Theorem 6.25].

Now let G be generated by elements x, y with defining relations

$$x^{p^m} = 1, \qquad y^{p^n} = x^{p^k}, \qquad y^{-1}xy = x^r \tag{1}$$

with suitable integers m, n, k and r [**3**, III.11.2]. The isomorphism of G and H we prove by induction on the order of $\langle x \rangle \cap \langle y \rangle$. So assume first that the generators x, y of G satisfy

$$x^{p^m} = 1, \quad y^{p^n} = 1, \quad y^{-1}xy = x^r, \tag{2}$$

that is $\langle x\rangle \cap \langle y\rangle = 1$. We proceed by induction on the order of G'. The case $|G'| = 1$ is known [**5**, 14.2.7ii]. Since H is metacyclic the isomorphism of G and H for $|G'| = p$ follows immediately from [**1**, Lemma 4 and Corollary of Theorem 2]. This is also the special case of [**6**, Theorem 6.25]. Let us assume now that $|G'| = p^s$, where $s > 1$. We have $\Omega_1(G') = G'^{(p^{s-1})}$, so by Lemma 3

$$\begin{aligned} F[G/\Omega_1(G')] &\simeq F[G]/(\omega(F[G'])F[G])^{p^{s-1}} \\ &\simeq F[H]/(\omega(F[H'])F[H])^{p^{s-1}} \simeq F[H/\Omega_1(H')] \end{aligned}$$

and by the induction $G/\Omega_1(G') \simeq H/\Omega_1(H')$. Since $G/\Omega_1(G')$ is generated by $\overline{x} = x\Omega_1(G')$, $\overline{y} = y\Omega_1(G')$ with relations

$$\overline{x}^{p^{m-1}} = 1, \quad \overline{y}^{p^n} = 1, \quad \overline{y}^{-1}\overline{x}\overline{y} = \overline{x}^r$$

there exist generators u, v of H such that

$$u^{p^{m-1}}, v^{p^n} \in \Omega_1(H'), \qquad v^{-1}uv \equiv u^r \pmod{\Omega_1(H')}.$$

Let $h \in \Omega_1(H')$ be such an element that $v^{-1}uv = u^r h$, and suppose $\Omega_1(H')$ is not a subgroup of $\langle u\rangle$. Then $(u,v) = u^{-1}v^{-1}uv = u^{r-1}h$ is a generator of H' and $(u,v)^p = (u^{r-1}h)^p = u^{p(r-1)}$. But H' is cyclic and $|H'| > p$ so that we have $\Omega_1(H') \le \langle (u,v)^p\rangle = \langle u^{p(r-1)}\rangle \le \langle u\rangle$. A contradiction. Thus $\Omega_1(H') = \langle u^{p^{m-1}}\rangle$ and $v^{p^n} \in \langle u^{p^{m-1}}\rangle$, $v^{-1}uv = u^{r+tp^{m-1}}$ for suitable t. Suppose that $v^{p^n} \ne 1$. If $n > m$, then $\exp(G) = p^n < p^{n+1} = \exp(H)$ which by Corollary 2 is impossible. If $n < m$ then one can choose an element u_1 of $\langle u\rangle$ such that $u_1^{p^n} = v^{p^n}$. Now replacing v by $v_1 = vu_1^{-1}$ we have $H = \langle u, v_1\rangle$, $v_1^{p^n} = 1$ and $v_1^{-1}uv_1 = v^{-1}uv$. The isomorphism of G and H follows now from [**2**, Lemma 8].

Suppose now that x and y are generators of G with the defining relations (1), the possible smallest order of $\langle x\rangle \cap \langle y\rangle$ and $\langle x\rangle \cap \langle y\rangle \ne 1$. Remark that the order of y must be greater than the order of x. Otherwise, using standard considerations one can replace y by an element y_1 such that $\langle x, y_1\rangle = G$ and $\langle x\rangle \cap \langle y_1\rangle = 1$. Assuming $G/\Omega_1(G') \simeq H/\Omega_1(H')$ as above we can choose generators u and v of H such that $\Omega_1(H') \le \langle u\rangle$ and

$$u^{p^m} = 1, \quad v^{-1}uv = u^{r+tp^{m-1}}, \quad v^{p^n} = u^{p^k + sp^{m-1}}$$

for suitable integers s, t. If $k < m-1$, then $p^k + sp^{m-1} \not\equiv 0 \pmod{p^m}$ and there exist an element $u_1 \in \langle u\rangle$ such that $\langle u\rangle = \langle u_1\rangle$, $v^{p^n} = u_1^{p^k}$ and $v^{-1}u_1v = u_1^{r+tp^{m-1}}$. So by [**2**, Lemma 8] $G \simeq H$. If $p^k + sp^{m-1} \equiv 0 \pmod{p^m}$, then $v^{p^n} = 1$ which implies $\exp(H) = p^n < \exp G$. By Corollary 2 it is impossible. This completes the proof.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, BIAŁYSTOK DIVISION, AKADEMICKA 2, 15-267 BIAŁYSTOK, POLAND