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NONCOMMUTATIVE REGULAR LOCAL RINGS OF DIMENSION 3

ROBERT L. SNIDER

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ABSTRACT. A left and right Noetherian local ring of global dimension 3 is a domain.

It is well known that a commutative Noetherian local ring of finite global dimension is an integral domain. We will say that a noncommutative ring R is local if R/J is a division ring where J is the Jacobson radical. Ramras asks if a noncommutative local Noetherian ring of finite global dimension is a (noncommutative) domain [**4**]. He showed that the answer is positive if the global dimension of R is at most 2 and R is both left and right Noetherian. If one assumes right Noetherian only, then a counterexample has been constructed by Stafford in dimension 2 [**2**, p. 138]. Positive answers for left and right Noetherian rings have been obtained by Brown, Hajarnavis, and MacEacharn for AR-rings of any finite dimension [**1**]. In this paper we improve on Ramras by showing that the answer is positive in dimension 3 for left and right Noetherian rings.

We make use of the notion of Krull dimension of a module as defined by Rentschler and Gabriel [**3**]. $C(N)$ will denote the elements of the ring R regular modulo the ideal N.

THEOREM. *A left and right Noetherian local ring R with global dimension* 3 *is a domain.*

PROOF. Let J be the maximal ideal of R and N the nil radical. By [**2**, p. 134], R/N is an integral domain and N is torsion with respect to $C(N)$. We suppose that N is nonzero. Let N be generated as a right ideal by $x_1, x_2, \ldots, x_n$. We may assume that this generating set is minimal. Now there exist $y_i \in C(N)$ such that $y_i x_i = 0$. Since R/N is left Noetherian R/N satisfies the left Ore condition by Goldie's theorem. Hence there exists $y \in C(N)$ such that $yx_i = 0$ for all i and hence $yN = 0$. Now the projective dimension of yR is at most 2. Since R/N is a domain the right annihilator of y is N. Hence the sequence $0 \to N \to R \to yR \to 0$ is exact. It follows that the projective dimension of N is at most 1. N cannot be projective, and therefore free, since N is $C(N)$ torsion. It follows that N has projective dimension 1. Define $f\colon R^n \to N$ by $f(r_1, \ldots, r_n) = \sum x_i r_i$. By the minimality of the x_i's, this is a projective cover of N. Let M be the kernel of f. This implies that $M \subseteq J^n$. Also M must be projective and hence free. Since N is

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$C(N)$ torsion, its reduced rank is 0 [**2**, p. 38]. Since the reduced rank is additive on short exact sequences, the reduced rank of M is n and M is free of rank n. Let $z_1, \ldots, z_n$ be a set of free generators for M. Let α be the minimum Krull dimension of nonzero right ideals of R. Let T be a right ideal maximal with respect to having Krull dimension α. T is a two-sided ideal. Suppose $A = \sum x_i T \neq 0$. Now the Krull dimension of A is α. Also $T^n/(T^n \cap M) \approx A$. Now $T^n \cap M$ is the largest submodule of M of Krull dimension α. Since M is free its largest submodule of Krull dimension α is $\sum z_i T \approx T^n$. Therefore T^n has a proper submodule isomorphic to T^n with factor module A with Krull dimension α. Iteration gives a proper descending chain all of whose factors are A. This contradicts the fact that Krull dimension of T^n is α. Therefore $A = 0$ and $T^n \subseteq M$. It follows that $T^n = \sum z_i T$. Projecting on the first factor we see that $T = JT$ since each $z_i \in J^n$. By Nakayama's lemma $T = 0$, a contradiction. Therefore $N = 0$ and hence R is a domain.

If R is any left and right Noetherian local ring of finite global dimension, then the above argument actually show that the projective dimension of any submodule of N is at least 2.

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Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061