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THE HODGE GROUP OF AN ABELIAN VARIETY

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ABSTRACT. Let A be a simple abelian variety of odd dimension, defined over $\mathbf{C}$. If the Hodge classes on A are intersections of divisors, then the semisimple part of the Hodge group of A is as large as it is allowed to be by endomorphisms and polarizations.

1. Introduction. Let A be an abelian variety defined over $\mathbf{C}$, and denote by $\mathscr{H}(A)$ its Hodge ring:

$$\mathscr{H}(A) = \bigoplus_p (H^{2p}(A(\mathbf{C}), \mathbf{Q}) \cap H^{pp}).$$

Denote by $\mathscr{D}(A)$ the subring of $\mathscr{H}(A)$ generated by the elements of $H^2(A(\mathbf{C}), \mathbf{Q}) \cap H^{11}$. We denote by $\mathrm{Hod}(A)$ the Hodge group of A introduced by Mumford [**3**]. It is a connected, reductive algebraic subgroup of $GL(V)$, where $V = H_1(A(\mathbf{C}), \mathbf{Q})$. It acts on the cohomology of A and its main property is that

$$H^*(A(\mathbf{C}), \mathbf{Q})^{\mathrm{Hod}(A)} = \mathscr{H}(A).$$

We shall also consider a group $L(A)$ which was studied by Ribet [**8**] and the author [**6**]. To define it, we fix a polarization ψ of A. Then, $L(A)$ is the connected component of the identity of the centralizer of $\mathrm{End}(A) \otimes \mathbf{Q}$ in $Sp(V, \psi)$. The definition is independent of the choice of polarization. It is known that $\mathrm{Hod}(A) \subseteq L(A)$ and in [**6**], it was proved that if A has no simple factor of type III (see §2 for the definitions), then

$$(*) \qquad \mathrm{Hod}(A) = L(A) \Leftrightarrow \mathscr{H}(A^k) = \mathscr{D}(A^k) \quad \text{for all } k \geq 1.$$

Moreover, if A has a factor of type III, then $\mathscr{H}(A) \neq \mathscr{D}(A)$.

Let G be a connected reductive algebraic group defined over $\mathbf{Q}$, and denote by $\mathscr{G}$ its Lie algebra. It is known that there is a unique connected semisimple algebraic subgroup G_{ss} of G whose Lie algebra is the maximal semisimple subalgebra of $\mathscr{G}$. We refer to G_{ss} as 'the semisimple part' of G.

The purpose of this note is to show that in some cases, the assumption $\mathscr{H}(A) = \mathscr{D}(A)$ is already sufficient to imply that the semisimple parts of $\mathrm{Hod}(A)$ and $L(A)$ are equal. We prove two general results of which the following is a consequence.

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PROPOSITION. *Let A be simple and of odd dimension. Then, $\mathscr{H}(A) = \mathscr{D}(A)$ implies that* $\mathrm{Hod}(A)_{\mathrm{ss}} = L(A)_{\mathrm{ss}}$.

A key element in the proof is the classification (cf. Serre [**10**]) of the simple factors of $\mathrm{Hod}(A)$ and the explicit determination of $L(A)$ [**6**].

REMARKS. 1. By a classical result of Lefschetz, $\mathscr{D}(A)$ consists of Poincaré duals of algebraic cycles on A. Thus, the condition $\mathscr{H}(A) = \mathscr{D}(A)$ implies that the Hodge conjecture holds for A. (In fact, it holds in the strong sense that any algebraic cycle on A is algebraically equivalent to an intersection of divisors.)

2. The proof will show that the condition $\mathscr{H}(A) = \mathscr{D}(A)$ can be replaced by the weaker condition that every Hodge cycle on A of codimension ≤ 3 is contained in $\mathscr{D}(A)$.

3. If we are in a case where $L(A)$ is known to be semisimple, the Proposition shows that $\mathscr{H}(A) = \mathscr{D}(A)$ implies that $\mathscr{H}(A^k) = \mathscr{D}(A^k)$ for all $k \geq 1$. This is because the conditions on A preclude it from being of type III and so $(*)$ can be applied.

4. If A is simple and of CM type, both $\mathrm{Hod}(A)$ and $L(A)$ are tori, and so the Proposition above gives no information. However, Lenstra and Ribet (unpublished) have shown in this case that if $\mathscr{H}(A) = \mathscr{D}(A)$ and $\mathrm{End}(A) \otimes \mathbf{Q}$ is a field which is *abelian* over $\mathbf{Q}$, then $\mathrm{Hod}(A) = L(A)$. It is an interesting problem to study how small $\mathrm{Hod}(A)$ can be, relative to $L(A)$, when A is of CM type. Liem Mai [**12**] has recently obtained *lower* bounds for the dimension of $\mathrm{Hod}(A)$, in some cases, and Dodson [**1**] has shown that this dimension must satisfy some congruence conditions.

5. Tanke′ev has shown that if A is simple and of prime dimension, $\mathrm{Hod}(A) = L(A)$ and $\mathscr{H}(A^k) = \mathscr{D}(A^k)$ for all $k \geq 1$. The proof has been simplified by Ribet [**8**] who also showed that the conclusion is valid for a slightly larger class of abelian varieties.

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2. Lemmas. Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let ρ denote complex conjugation. We write

$$\Delta(A) = \mathrm{End}(A) \otimes_Z \mathbf{Q}$$

and let E be a maximal commutative semisimple subalgebra of $\Delta(A)$. Thus, E is a product of fields $E = E_1 \times \cdots \times E_r$ and by the classification of Albert (cf. Mumford [**4**, §21]) each factor is either a CM field or a totally real field. We say that A is of type III if $\Delta(A)$ is a totally definite quaternion division algebra over a totally real field, and A is of type IV if $\Delta(A)$ is a division algebra over a CM field.

We have a decomposition $V \otimes \mathbf{C} = \prod V_\sigma$ indexed by the set Σ of homomorphisms $\sigma\colon E \otimes \mathbf{C} \to \mathbf{C}$, where

$$V_\sigma = (V \otimes \mathbf{C}) \otimes_{E \otimes \mathbf{C}, \sigma} \mathbf{C}.$$

Let $H = \mathrm{Hod}(A)_{/\mathbf{C}}$ and $L = L(A)_{/\mathbf{C}}$. Each V_σ is an L module and hence, also an H module. Denote by L_σ the projection of L to $GL(V_\sigma)$.

LEMMA 1. *Each V_σ is a simple H module, and the following are equivalent:*
(i) $V_{\sigma_1} \sim V_{\sigma_2}$ *as H modules,*
(ii) $V_{\sigma_1} \sim V_{\sigma_2}$ *as L modules.*

PROOF. It follows easily from the definitions that

$$\mathrm{End}_{\mathrm{Hod}(A)}(V) = \mathrm{End}_{L(A)}V = \Delta(A).$$

As E is a maximal commutative semisimple subalgebra, we have

$$\mathrm{End}_{\mathrm{Hod}(A),E}V = E.$$

Complexifying, we see that for any $\sigma \in \Sigma$, we have

$$\mathrm{End}_H V_\sigma = \mathbf{C}.$$

This proves the first assertion. To see the equivalence of (i) and (ii), note that for any $\sigma_1, \sigma_2 \in \Sigma$, we have $\mathrm{Hom}_L(V_{\sigma_1}, V_{\sigma_2}) \hookrightarrow \mathrm{Hom}_H(V_{\sigma_1}, V_{\sigma_2})$. Now

$$\prod \mathrm{Hom}_H(V_{\sigma_1}, V_{\sigma_2}) = \prod \mathrm{Hom}_L(V_{\sigma_1}, V_{\sigma_2})$$

as both sides are equal to $\Delta(A) \otimes \mathbf{C}$. (The product on both sides is over all pairs $\sigma_1, \sigma_2 \in \Sigma$.) The result follows. □

Denote by E_i^+ the maximal totally real subfield of E_i $(1 \le i \le r)$ and set $E^+ = E_1^+ \times \cdots \times E_r^+$. For each homomorphism $\lambda\colon E^+ \otimes \mathbf{R} \to \mathbf{R}$, set

$$X_\lambda = (V \otimes \mathbf{R}) \otimes_{E^+ \otimes \mathbf{R}, \lambda} \mathbf{R}.$$

Since $L(A)$ commutes with E, it acts on X_λ. We write L_λ for the projection $L(A) \to GL(X_\lambda)$.

Suppose that every simple factor of A is of type IV. Then, for every λ, X_λ has a natural complex structure and the polarization induces a Hermitian form ω_λ on X_λ. Denote by $U(X_\lambda, \omega_\lambda)$ the unitary group with respect to ω_λ.

LEMMA 2. *If every simple factor of A is of type* IV, *then for every λ, $L_\lambda = U(X_\lambda, \omega_\lambda)$.*

PROOF. This follows from Lemma 2.1 and Lemma 2.3 of [**6**]. □

We know that

$$H \subseteq L \subseteq \prod GL(V_\sigma).$$

Let $S = H_{\mathrm{ss}}$ and $R = L_{\mathrm{ss}}$. Let H_σ denote the projection of H to $GL(V_\sigma)$, and S_σ the projection of S to $SL(V_\sigma)$.

LEMMA 3. *If V_σ is of odd dimension (over* $\mathbf{C}$*), then every simple component of S_σ is of type* $\mathbf{A}$. *Moreover, if $V_\sigma = V_{1,\sigma} \otimes V_{2,\sigma} \otimes \cdots \otimes V_{m,\sigma}$ is a decomposition of V_σ induced by the decomposition of S_σ into simple components, each $V_{i,\sigma}$ is an exterior power of the standard representation (i.e. a fundamental representation).*

This is [**10**, Corollary to Proposition 8].

LEMMA 4. *Let S_i $(1 \le i \le r)$ be complex simple Lie groups of type $\mathbf{A}_{m(i)}$, $m(i) \ge 3$ for all i. Let $Y_1, \ldots, Y_r$ be fundamental representations of $S_1, \ldots, S_r$ (respectively). Let*

$$S = S_1 \times S_2 \times \cdots \times S_r \quad \textit{and} \quad Y = Y_1 \otimes Y_2 \otimes \cdots \otimes Y_r.$$

If the S modules $\bigwedge^2 Y$ and $\bigwedge^3 Y$ are irreducible, then $r = 1$ and Y is the standard representation of S or its dual.

PROOF. For each i,

$$\mathrm{Sym}^2(Y_1) \otimes \cdots \otimes \mathrm{Sym}^2(Y_{i-1}) \otimes \bigwedge^2 Y_i \otimes \mathrm{Sym}^2(Y_{i+1}) \otimes \cdots \otimes \mathrm{Sym}^2(Y_r)$$

is an S submodule of $\bigwedge^2 Y$. Thus, $\bigwedge^2 Y$ is reducible unless $r = 1$. Now, let W denote the standard representation of S and set $w = \dim W$. Suppose that $Y = \bigwedge^j W$. We may assume that $j \leq w/2$ by replacing Y with its dual if necessary. In the notation of Jacobson [**2**, Chapters 4 and 8], the highest weight of the exterior square of Y is $\alpha_j + \alpha_{j+1}$. By the Weyl dimension formula, we must then have

$$\binom{\binom{w}{j}}{2} = \dim \bigwedge^2 Y = \frac{3(w+1)}{(j+1)(w-j+2)} \binom{w}{j-1} \binom{w}{j+1}.$$

Easy estimates show that this is possible only if $j \leq 2$. If $j = 2$, the highest weight of the exterior cube of Y is $2\alpha_1 + \alpha_4$ and again, by the dimension formula, we must have

$$\binom{\binom{w}{2}}{3} = \dim \bigwedge^3 Y = \frac{1}{3}(w+1)(w+2) \binom{w}{4}$$

which forces $w \leq 2$. Thus, we must have $j = 1$ and the lemma is proved.

LEMMA 5. *Let W_1, W_2 be two finite dimensional complex vector spaces. Let $\mathscr{S}_1, \mathscr{S}_2$ be simple complex Lie subalgebras of $\mathscr{GL}(W_1)$, $\mathscr{GL}(W_2)$ (respectively) of type* **A**, **B** *or* **C**. *Let $\mathscr{S}$ be a Lie subalgebra of $\mathscr{S}_1 \times \mathscr{S}_2$ whose projection to each factor is surjective. Then, either $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$ or $\mathscr{S}$ is the graph of an isomorphism $\mathscr{S}_1 \simeq \mathscr{S}_2$ induced by an isomorphism $W_2 \simeq W_1$ or $\check{W}_1$ of $\mathscr{S}$-modules.*

PROOF. For $i = 1, 2$, let $\pi_i \colon \mathscr{S} \to \mathscr{S}_i$ denote the ith projection and set $\mathscr{N}_i = \ker \pi_{3-i}$. We may view $\mathscr{N}_i$ as an ideal of $\mathscr{S}_i$ and hence, $\mathscr{N}_i = 0$. By Goursat's Lemma (see Ribet [**7**, §5]), it follows that $\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2$ or $\mathscr{S}$ is the graph of an isomorphism $\mathscr{S}_1 \simeq \mathscr{S}_2$. In the latter case, it follows that $\mathscr{S}_1, \mathscr{S}_2$ are both of the same type. If they are of type **B** or **C**, it follows that $W_1 \simeq W_2$ as $\mathscr{S}$-modules, since every automorphism of such algebras is inner [**2**, p. 281]. If they are of type **A**, then $W_1 \simeq W_2$ or $\check{W}_2$ as $\mathscr{S}$ modules, since, modulo inner automorphisms, the only automorphism of such an algebra is $g \mapsto -{}^t g$.

LEMMA 6. *Let I be a finite set and for each $\sigma \in I$, let $\mathscr{S}_\sigma$ be a finite dimensional complex simple Lie algebra. Let $\mathscr{A}, \mathscr{B}$ be two algebras such that*

(a) $\mathscr{A} \subseteq \mathscr{B}$.

(b) *$\mathscr{B}$ is a subalgebra of $\prod \mathscr{S}_\sigma$ and the projection to each $\mathscr{S}_\sigma$ is surjective.*

(c) *$\mathscr{A}, \mathscr{B}$ have equal images on $\mathscr{S}_\sigma \times \mathscr{S}_\tau$ for all pairs $(\sigma, \tau) \in I \times I$, $\sigma \neq \tau$. Then, $\mathscr{A} = \mathscr{B} = \prod \mathscr{S}_\sigma$ where the σ range over a certain subset of I.*

This is [**9**, Lemma 4.6].

3. Main results. We retain the notation of the previous section. Thus, A is an abelian variety, $V = H_1(A(\mathbf{C}), \mathbf{Q})$ and E is a maximal commutative semisimple subalgebra of $\Delta(A)$.

THEOREM 1. *Suppose that E is a product of CM fields and that V is free over E of odd rank m. Suppose that $\mathscr{H}(A) = \mathscr{D}(A)$. Then* $\mathrm{Hod}(A)_{\mathrm{ss}} = L(A)_{\mathrm{ss}}$.

PROOF. First, we check that every simple factor of A is of type IV. We have $E = E_1 \times \cdots \times E_r$, with E_i a CM field. There is a corresponding decomposition up to isogeny $A \sim A_1 \times \cdots \times A_r$. As E_i is a field, A_i is a power of a simple abelian variety, say $A_i = B_i^{n(i)}$. To show that B_i is of type IV, it suffices to show that $\mathrm{End}(B_i) \otimes \mathbf{Q}$ is not a quaternion division algebra D_i over a totally real field F_i. Suppose that it is and let $x_i = \dim_{D_i} H_1(B_i(\mathbf{C}), \mathbf{Q})$. As E_i is a maximal commutative subfield of

$$\Delta(A_i) = M_{n(i)}(D_i),$$

we have

$$m = \frac{dimV_i}{[E_i : \mathbf{Q}]} = \frac{n(i)x_i[D_i : \mathbf{Q}]}{2n(i)[F_i : \mathbf{Q}]} = 2x_i.$$

This contradicts our assumption that m is odd. Thus, every simple factor of A is of type IV.

Now, for each homomorphism $\lambda\colon E^+ \otimes \mathbf{R} \to \mathbf{R}$, we have

$$X_\lambda \otimes \mathbf{C} = V_\sigma \oplus V_{\rho\sigma}$$

where σ is an extension of λ to a map $E \otimes \mathbf{C} \to \mathbf{C}$. Lemma 2 implies that as L modules,

$$V_{\rho\sigma} \simeq \check{V}_\sigma. \tag{1}$$

Moreover, Lemma 1 implies that the V_σ are all simple as H modules. Let R_σ denote the image of R in $SL(V_\sigma)$. By Lemma 2, $R_\sigma = SL(V_\sigma)$. As $\dim_{\mathbf{C}}(V_\sigma) = m$ is odd, $m = 1$ or $m \geq 3$. In the first case, $S_\sigma = R_\sigma = 1$. We may thus suppose that $m \geq 3$. By Lemma 3, there is an isogeny $S_\sigma \sim S_{1,\sigma} \times \cdots \times S_{r,\sigma}$ with each $S_{i,\sigma}$ a simple group of type $\mathbf{A}$. In the corresponding decomposition $V_\sigma = V_{1,\sigma} \otimes \cdots \otimes V_{r,\sigma}$, each $V_{i,\sigma}$ is a fundamental representation of $S_{i,\sigma}$.

By [**6**, Lemmas 2.1, 3.4 and 3.6] and our assumption that $\mathscr{H}(A) = \mathscr{D}(A)$, we see that

$$H^*(A(\mathbf{C}), \mathbf{Q})^{\mathrm{Hod}(A)} = \mathscr{H}(A) = \mathscr{D}(A) = H^*(A(\mathbf{C}), \mathbf{Q})^{L(A)}.$$

Thus, for any sequence $\{i(\sigma)\}_\sigma$ of positive integers,

(**) *the H and L invariants of* $\bigotimes_\sigma \bigwedge^{i(\sigma)} \check{V}_\sigma$ *must agree.*

In particular, utilising (1), we see that for any σ, and any integer i with $1 \leq i \leq m$, we must have

$$\dim \mathrm{End}_{H_\sigma}\left(\bigwedge^i V_\sigma\right) = \dim \mathrm{End}_{L_\sigma}\left(\bigwedge^i V_\sigma\right) = 1. \tag{2}$$

Since $H_\sigma \subseteq L_\sigma = GL(V_\sigma)$, the center Z_σ of H_σ consists at most of scalars. Thus (2) is equivalent to

$$\mathrm{End}_{S_\sigma}\left(\bigwedge^i V_\sigma\right) = \mathbf{C} \tag{3}$$

for all σ. In particular, $\bigwedge^2 V_\sigma$ and $\bigwedge^3 V_\sigma$ are irreducible as S_σ modules. Since $S_\sigma \subseteq R_\sigma$ and R_σ acts on V_σ by the standard representation, Lemma 4 implies that

$$S_\sigma = SL(V_\sigma) = R_\sigma.$$

In particular, the Lie algebra $\mathscr{S}_\sigma$ of S_σ is $\mathscr{SL}(V_\sigma)$. Now, for $\sigma, \tau \in \Sigma$, $\sigma \neq \tau$, let $S_{\sigma,\tau}$ (respectively, $R_{\sigma,\tau}$) denote the image of S (respectively, R) in $SL(V_\sigma) \times SL(V_\tau)$. Let $\mathscr{S}_{\sigma,\tau}$ (respectively, $\mathscr{R}_{\sigma,\tau}$) denote the Lie algebra of $S_{\sigma,\tau}$ (respectively, $R_{\sigma,\tau}$). Goursat's Lemma implies that

$$\mathscr{S}_{\sigma,\tau} = \mathscr{SL}(V_\sigma) \times \mathscr{SL}(V_\tau) \tag{4}$$

or $\mathscr{S}_{\sigma,\tau}$ is the graph of an isomorphism

$$\mathscr{SL}(V_\sigma) \simeq \mathscr{SL}(V_\tau). \tag{5}$$

If (4) occurs, then we see that

$$\mathscr{R}_{\sigma,\tau} = \mathscr{SL}(V_\sigma) \times \mathscr{SL}(V_\tau)$$

also, since $\mathscr{S}_{\sigma,\tau} \subseteq \mathscr{R}_{\sigma,\tau}$.

Now suppose (5) occurs. Lemma 5 and (1) imply that

$$V_\sigma \simeq V_\tau \ \ or \ V_{\rho\tau} \tag{6}$$

as S modules. If $\tau = \rho\sigma$, then it is clear that $\mathscr{R}_{\sigma,\tau}$ is also the graph of the isomorphism (5). We may therefore suppose that τ is different from $\rho\sigma$. Consider the H submodule of $H_4(A(\mathbf{C}), \mathbf{C})$ given by

$$W = V_\sigma \otimes V_{\rho\sigma} \otimes V_\tau \otimes V_{\rho\tau}. \tag{7}$$

From (6), it follows that as S modules,

$$\begin{aligned} W &\simeq V_\sigma \otimes V_\sigma \otimes V_{\rho\sigma} \otimes V_{\rho\sigma} \simeq \mathrm{End}(V_\sigma \otimes V_\sigma) \\ &\simeq \mathrm{End}\left(\bigwedge^2 V_\sigma \oplus \mathrm{Sym}^2 V_\sigma\right). \end{aligned}$$

The latter contains a two dimensional subspace

$$W_{\sigma,\tau} = \mathrm{End}_S\left(\bigwedge^2 V_\sigma\right) \oplus \mathrm{End}_S(\mathrm{Sym}^2 V_\sigma)$$

on which S acts trivially. The center of H acts trivially on W, and hence also on $W_{\sigma,\tau}$. It follows that H acts trivially on $W_{\sigma,\tau}$. By our assumption (**), L also acts trivially on this space. It follows that (6) holds as L modules also, for otherwise, the subspace of W on which L acts trivially is only one dimensional. Hence, we deduce that $\mathscr{R}_{\sigma,\tau}$ is the graph of an isomorphism $\mathscr{SL}(V_\sigma) \simeq \mathscr{SL}(V_\tau)$ and in particular, $\mathscr{R}_{\sigma,\tau} = \mathscr{S}_{\sigma,\tau}$.

Now, appealing to Lemma 6, we conclude that

$$\mathscr{S} = \mathscr{R} = \prod \mathscr{S}\mathscr{U}(X_\lambda)$$

where λ ranges over a certain subset of the homomorphisms $E^+ \otimes \mathbf{R} \to \mathbf{R}$. Since S and R are connected, and $S \subseteq R$, it follows that $S = R$. This proves the Theorem.

REMARK. In fact, it can be shown that $H_\sigma = GL(V_\sigma)$ and that the dimension of the center of H is at least 2. Hence, if $\dim_{\mathbf{Q}} E \leq 4$, then $\mathrm{Hod}(A) = L(A)$.

The next theorem is a slight generalization of a result of Tanke′ev [**11**] who proved it in the case $\Delta(A) = E$. The case $m = 1$ is a special case of [**5**, Theorem 4.1].

THEOREM 2. *Suppose that E is a product of totally real fields and that V is free over E of rank $2m$, m odd. Then* $\mathrm{Hod}(A) = L(A)$. *In particular, $\mathscr{H}(A^k) = \mathscr{D}(A^k)$ for all $k \geq 1$.*

PROOF. As before, for each homomorphism $\sigma\colon E \otimes \mathbf{C} \to \mathbf{C}$, we have the H module V_σ which is a $2m$ dimensional $\mathbf{C}$ vector space. Write $H_\sigma \sim H_{1,\sigma} \times \cdots \times H_{r,\sigma}$ where each $H_{i,\sigma}$ is a simple group. Write $V_\sigma = V_{1,\sigma} \otimes \cdots \otimes V_{r,\sigma}$ for the corresponding decomposition of V_σ. By Lemma 1, each V_σ is a simple H_σ module. As E is a product of totally real fields, the polarization ψ induces an alternating form ψ_σ on V_σ and $H_\sigma \subseteq L_\sigma \subseteq Sp(V_\sigma, \psi_\sigma)$. (Here, L_σ is the projection of L to $GL(V_\sigma)$.) Hence, each $V_{i,\sigma}$ is either a symplectic or orthogonal representation of $H_{i,\sigma}$. Since a symplectic representation has even dimension, exactly one of the $V_{i,\sigma}$ is symplectic and the others must be orthogonal. From [**10**, Proposition 7 and the Appendix], we see that when m is odd, there are the following possibilities:

(i) $H_{i,\sigma}$ is of type $\mathbf{A}_n$ $(n \geq 1)$ and $V_{i,\sigma}$ is an exterior power of the standard representation. Here, $\dim V_{i,\sigma} = \binom{n+1}{k}$ for some $k \leq n$.

(ii) $H_{i,\sigma}$ is of type $\mathbf{C}_n$ $(n \geq 2)$ and $V_{i,\sigma}$ is the standard representation. Here, $\dim V_{i,\sigma} = 2n$ and $V_{i,\sigma}$ is symplectic.

(iii) $H_{i,\sigma}$ is of type $\mathbf{D}_n$ $(n \geq 4)$ and $V_{i,\sigma}$ is the standard representation. Here, $\dim V_{i,\sigma} = 2n$ and $V_{i,\sigma}$ is orthogonal.

Suppose $H_{i,\sigma}$ is of type $\mathbf{A}_n$. For $V_{i,\sigma}$ to be symplectic or orthogonal, we must have n odd, say $n+1 = 2n_0$, and $k = n_0$. Moreover, $V_{i,\sigma}$ is symplectic when n_0 is odd and orthogonal when n_0 is even. Now,

$$\dim V_{i,\sigma} = \binom{n+1}{k} = \binom{2n_0}{n_0}$$

is always even and is divisible by 4 if n_0 is odd. Thus, m odd implies that $V_{i,\sigma}$ cannot be symplectic. But in this case, some other $V_{j,\sigma}$ is symplectic and this would give $4 \mid 2m$ and so, case (i) cannot occur. But, as $\dim V_{i,\sigma}$ is even in both the remaining cases, we must have $r = 1$ and H_σ of type $\mathbf{C}_n$, i.e. $H_\sigma = Sp(V_\sigma, \psi_\sigma) = L_\sigma$. Again, by taking Lie algebras and appealing to Lemmas 1, 5 and 6, it follows that

$$H = L = \prod Sp(V_\sigma, \psi_\sigma).$$

Finally, we check that no simple factor of A is of type III by a dimension calculation as in the proof of Theorem 1. Now by [**6**, Theorem 3.1] (stated as $(*)$ in the Introduction), it follows that $\mathscr{H}(A^k) = \mathscr{D}(A^k)$ for all $k \geq 1$. This proves the theorem.

Finally, we combine the two theorems to prove the result stated in the Introduction.

PROOF OF THE PROPOSITION. As A is simple, any maximal commutative semisimple subalgebra of $\Delta(A)$ must in fact be a field E, (say). Moreover, E is totally real or a CM field. In the first case, E acts on $\mathrm{Lie}(A)$ and so $[E : \mathbf{Q}]$ divides $\dim A$. Thus, $\dim_E V = 2m$, m odd. The Proposition follows in this case from Theorem 2. In the second case, $[E : \mathbf{Q}]$ is even and so $\dim_E V = m$ is odd. The Proposition follows in this case from Theorem 1.

REFERENCES

1. B. Dodson, *On the Mumford-Tate group of an abelian variety with complex multiplication*, J. Algebra **111** (1987), 49–73.
2. N. Jacobson, *Lie algebras*, Interscience, 1962.
3. D. Mumford, *Families of abelian varieties*, Algebraic Groups and Discontinuous Subgroups (A. Borel and G. Mostow, eds.), Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R. I., 1966, pp. 347–351.
4. ——, *Abelian varieties*, Tata Institute Lecture Notes, Oxford, 1974.
5. V. Kumar Murty, *Algebraic cycles on abelian varieties*, Duke Math. J. **50** (1983), 487–504.
6. ——, *Exceptional Hodge classes on certain abelian varieties*, Math. Ann. **268** (1984), 197–206.
7. K. Ribet, *Galois action on division points of abelian varieties with real multiplication*, Amer. J. Math. **98** (1976), 751–804.
8. ——, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. **105** (1983), 523–538.
9. ——, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. **253** (1980), 43–62.
10. J.-P. Serre, *Groupes algébriques associés aux modules de Hodge-Tate*, Astérisque **65** (1979), 155–188.
11. S. G. Tanke′ev, *On algebraic cycles on surfaces and abelian varieties*, Math. USSR Izv. **18** (1982), 349–380.
12. Liem Mai, *Lower bounds for the rank of a CM type*, M.Sc. thesis, Concordia University, 1987.

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