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FRÉCHET DIFFERENTIABLE POINTS IN BOCHNER FUNCTION SPACES $L_p(\mu, X)$

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ABSTRACT. In this paper, a characterization of Fréchet differentiable points of $L_p(\mu, X)$, $1 < p < \infty$, is given: $f \in L_p(\mu, X)$, $f \neq 0$ is a point of Fréchet differentiability of the norm if and only if the values $f(t)$ are such almost everywhere in the support of f.

Let X be a Banach space, (Ω, Σ, μ) be a measure space, $L_p(\mu, X)$ denotes the Lebesgue-Bochner space of equivalence classes of p-integrable, X-valued, strongly measurable functions on the measure space (Ω, Σ, μ). The norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = \left(\int_\Omega \|f(t)\|_X^p \, d\mu\right)^{1/p}, \qquad f \in L_p(\mu, X).$$

In [1], Greim proved that if $g \in L_p(\mu, X)^*$, then g strongly exposes f ($f \neq 0$) if and only if $|g|$ strongly exposes $|f|$ and for almost all $t \in \Omega$, $g(t)$ strongly exposes $f(t)$ or $g(t) = 0 = f(t)$. Using his result, we prove that the natural conditions are sufficient and necessary for f to be a Fréchet differentiable point of $L_p(\mu, X)$. Consequently, it follows that $L_p(\mu, X)$ is Fréchet differentiable if and only if X is Fréchet differentiable. This result is due to Leonard and Sundaresan [**2**].

DEFINITION 1. We say that the norm of the Banach space X is Fréchet differentiable at $x_0 \neq 0$ whenever

$$\lim_{\lambda \to 0} \frac{\|x_0 + \lambda y\| - \|x_0\|}{\lambda}$$

exists uniformly for $y \in S(X) = \{x\colon \|x\| = 1\}$. If the norm of X is Fréchet differentiable at each $x \neq 0$ then we say that X has a Fréchet differentiable norm.

In this paper, we say that $x \neq 0$ is a Fréchet differentiable point of the Banach space X if the norm of X is Fréchet differentiable at x.

DEFINITION 2. An element x of a normed space X is said to be strongly exposed by an element x^* of the dual X^* if

(i) $x^*(x) = \|x^*\| \cdot \|x\| \neq 0$, and

(ii) each sequence (x_n) in the ball with radius $\|x\|$, such that $x^*(x_n) \to x^*(x)$, converges to x in norm.

For arbitrary functions $f\colon \Omega \to X$ and $g\colon \Omega \to X^*$ let us denote the functions $t \mapsto \|f(t)\|$, $\|g(t)\|$ by $|f|$, $|g|$, respectively.

Obviously, if $f \in L_p(\mu, X)$, $g \in L_p(\mu, X)^*$ and $|g|/\|g\| = (|f|/\|f\|_p)^{p-1}$, then $|g|$ strongly exposes $|f|$, $1 < p < \infty$.

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LEMMA 1. *Any one of the following statements implies all the others.*

(i) *The norm is Fréchet differentiable at* x_0.

(ii) *Any support mapping* $x\ (\neq 0) \to$ *norm* 1 *functional supporting the unit ball in* $x/\|x\|$ *is norm-norm-continuous at* x_0.

(iii) $x_0 \neq 0$ *strongly exposes some* $x_0^* \in X^*$ *with* $\|x_0^*\| = 1$.

The equivalence of (i) and (ii) is well known (see [**3**]). The equivalence of (ii) and (iii) is clear.

LEMMA 2 [**1**]. *Let* (Ω, Σ, μ) *be a positive measure space,* X *be a Banach space,* $1 < p < \infty$, *and* $f \in L_p(\mu, X)$, $g \in L_p(\mu, X)^*$. *Then* g *strongly exposes* f *if and only if* $|g|$ *strongly exposes* $|f|$ *and for almost all* $t \in \Omega$, $g(t)$ *strongly exposes* $f(t)$ *or* $g(t) = 0 = f(t)$.

THEOREM 1. *Let* (Ω, Σ, μ) *be a positive measure space,* X *be a Banach space,* $1 < p < \infty$. *If* $f \in L_p(\mu, X)$, $f \neq 0$, *and for almost all* t *in* $\operatorname{supp}(f) = \{t \in \Omega\colon f(t) \neq 0\}$, $f(t)$ *is a Fréchet differentiable point of* X, *then* f *is a Fréchet differentiable point of* $L_p(\mu, X)$.

PROOF. We can assume that for each $t \in \operatorname{supp}(f)$, $f(t)$ is a Fréchet differentiable point of X. By Lemma 1(iii), $f(t)$ strongly exposes some element x_t^* in X^* with $\|x_t^*\| = 1$. Define

$$g_0(t) = \begin{cases} x_t^*, & t \in \operatorname{supp}(f), \\ 0, & t \in \Omega \backslash \operatorname{supp}(f). \end{cases}$$

Let (f_n) be a sequence of simple functions in $L_p(\mu, X)$, such that $f_n(t) \to f(t)$, a.e. We can assume without loss of generality that $f_n(t) \neq 0$ if $t \in \operatorname{supp}(f)$; $f_n(t) = 0$ if t in $\Omega\backslash\operatorname{supp}(f)$. For each $t \in \operatorname{supp}(f)$, observe that $f(t)$ is a Fréchet differentiable point of X. Let $x_{n,t}^*$ be a support functional of $f_n(t)/\|f_n(t)\|$; by Lemma 1(ii), it follows that, $\|x_{n,t}^* - g_0(t)\| \to 0$, $n \to \infty$, for almost all t in $\operatorname{supp}(f)$. For simple function f_n, if t, $t' \in \operatorname{supp}(f)$ and $f_n(t) = f_n(t')$, take $x_{n,t}^* = x_{n,t'}^*$; if $t \in \Omega \backslash \operatorname{supp}(f)$, take $x_{n,t}^* = 0$. Then $(x_{n,t}^*)$ is a sequence of simple functions, and

$$\|x_{n,t}^* - g_0(t)\| \to 0, \qquad t \in \Omega, \quad \text{a.e.}$$

This shows that $g_0(t)$ is a μ measurable function. Define

$$g(t) = \|f(t)\|^{p-1} g_0(t), \qquad t \in \Omega.$$

Obviously, $g(t)$ is a μ measurable function and $\int_\Omega \|g(t)\|^q \, d\mu = \|f\|_p^p$. Therefore $g \in L_q(\mu, X^*)$ and $\|g\|_q^q = \|f\|_p^p$. Also

$$\|g(t)\| = \|f(t)\|^{(p-1)} \|g_0(t)\| = \|f(t)\|^{(p-1)},$$

therefore,

$$\|g(t)\|^{(q-1)} = \|f(t)\|^{(p-1)(q-1)} = \|f(t)\|,$$

thus

$$\|f(t)\|/\|f\|_p = (\|g(t)\|/\|g\|_q)^{(q-1)}, \qquad t \in \Omega.$$

Consequently,

$$|f|/\|f\|_p = (|g|/\|g\|_q)^{(q-1)}.$$

It is shown that $|f|$ strongly exposes $|g|$. Since $f(t)$ strongly exposes $g_0(t)$, if $t \in \operatorname{supp}(f)$, hence $f(t)$ strongly exposes $g(t)$ or $f(t) = 0 = g(t)$. By Lemma 2, f

strongly exposes g. Straight away from the Lemma 1(iii), we conclude that f is a Fréchet differentiable point of $L_p(\mu, X)$.

COROLLARY [**2**]. *If the norm of X is Fréchet differentiable, then the norm of $L_p(\mu, X)$ is Fréchet differentiable.*

THEOREM 2. *Let (Ω, Σ, μ) be a positive measure space, X be a Banach space, $1 < p < \infty$, $f \in L_p(\mu, X)$, $f \neq 0$. If f is a Fréchet differentiable point of $L_p(\mu, X)$, then for almost all t in* supp(f), *$f(t)$ is a Fréchet differentiable point of X.*

PROOF. Since f is a Fréchet differentiable point of $L_p(\mu, X)$, by Lemma 1(iii), f strongly exposes an element g in $L_p(\mu, X)^*$ with $\|g\| = 1$. We now prove that g is in $L_q(\mu, X^*) \subset L_p(\mu, X)^*$. In fact, we may choose a sequence of simple functions (f_n) in $L_p(\mu, X)$, such that $\|f_n - f\|_p \to 0$. Observe that for every simple function f_n, $f_n/\|f_n\|_p$ is supported by a simple function g_n in $L_q(\mu, X^*)$ with $\|g_n\|_q = 1$. By Lemma 1(ii), $\|g_n - g\| \to 0$, hence g is in $L_q(\mu, X^*)$. By Lemma 2, for almost all $t \in \Omega$, $f(t)$ strongly exposes $g(t)$ or $f(t) = 0 = g(t)$. Obviously, for almost all t in supp(f), $f(t)$ strongly exposes $g(t)/\|g(t)\|$; by Lemma 1(iii), it follows that $f(t)$ is a Fréchet differentiable point of X, for almost all t in supp(f).

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