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GENERALIZED OPEN MAPPING THEOREMS FOR BILINEAR MAPS, WITH AN APPLICATION TO OPERATOR ALGEBRAS

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ABSTRACT. Cohen [**4**] gave an example of a surjective bilinear mapping between Banach spaces which was not open, and Horowitz [**8**] gave a much simpler example. We build on Horowitz' example to produce a similar result for bilinear mappings such that every element of the target space is a linear combination of n elements of the range. An immediate application is that Bercovici's construction [**1**] of an operator algebra with property $(\mathbf{A}_1)$ but not $(\mathbf{A}_1(r))$ can be extended to achieve property $(\mathbf{A}_{1/n})$ without $(\mathbf{A}_{1/n}(r))$.

Let $T\colon X \times Y \to Z$ be a continuous bilinear mapping between Banach spaces. We shall be concerned with the following two properties of such a mapping.

DEFINITION. For $n = 1, 2, 3, \ldots$ we say T is $(1/n)$-*surjective* if for every $z \in Z$ there exist $x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_n \in Y$ with

$$z = T(x_1, y_1) + T(x_2, y_2) + \cdots + T(x_n, y_n). \tag{1}$$

If, further, there exists a constant $K > 0$ such that for every z the $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ may be found, satisfying (1), with

$$\sum_{i=1}^{n} \|x_i\|\|y_i\| \le K\|z\|,$$

then we say that T is $(1/n)$-*open*.

REMARKS. (a) These properties are generalizations of the properties $(\mathbf{A}_{1/n})$ and "$(\mathbf{A}_{1/n}(r))$ for some $r > 0$" of operator algebras studied by Bercovici, Foiaş and Pearcy [**2**]: hence the "$1/n$" in our notation.

(b) If $n = 1$, "$(1/n)$-surjective" is "surjective" and "$(1/n)$-open" is "open".

(c) Notice that

$$\sum_{i=1}^{n} T(x_i, y_i) = \sum_{i=1}^{n} T(\lambda_i x_i, \lambda_i^{-1} y_i)$$

for any nonzero scalars $\lambda_1, \ldots, \lambda_n$. We may therefore arrange that $\|x_i\| = \|y_i\|$ $(1 \le i \le n)$. Thus, T is $(1/n)$-open if and only if there is a constant $L > 0$ such that every $z \in Z$ with $\|z\| \le 1$ may be expressed in the form (1) with $\|x_i\| \le L$, $\|y_i\| \le L$ $(1 \le i \le n)$.

(d) We may work over the real field or the complex field and, in the latter case, we may study either bilinear or sesquilinear maps. The definitions above and Theorems 1 and 2 below apply to all three cases.

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The best positive result known is the following, which is present, at least implicitly, in [**9**, Theorem 1.3, **7**, Lemma 3.5, **3** and **5**].

THEOREM 1. *Suppose $T\colon X \times Y \to Z$ is a continuous bilinear (or sesquilinear) map between separable Banach spaces X, Y, Z. Then*

(a) *if, for every $z \in Z$, there exists a positive integer n and $x_1, \dots, x_n \in X$, $y_1, \dots, y_n \in Y$ satisfying* (1), *then T is $(1/n)$-surjective for some n (i.e. the dependence of n on z can be removed);*

(b) *if T is $(1/n)$-surjective, then T is $(1/2n)$-open.*

It is known [**6**] that the separability condition cannot be removed from (a). We conjecture that it cannot be removed from (b) either. Horowitz [**8**] showed that, even for finite-dimensional X, Y, Z, (b) cannot be improved to "surjective implies open" in the $n = 1$ case. Our main result here is the extension of his example to show that, generally, $(1/n)$-surjective does not imply $(1/n)$-open. It remains an interesting open question whether $(1/n)$-surjective implies $(1/m)$-open for $n < m < 2n$.

THEOREM 2. *For each positive integer n, there exists a $(1/n)$-surjective sesquilinear (or bilinear) map $T\colon \mathbb{C}^{n+2} \times \mathbb{C}^{n+2} \to \mathbb{C}^N$ (where $N = n^2 + n + 2$), which is not $(1/n)$-open.*

PROOF. We show the existence of a sesquilinear map T: a complex bilinear map $\tilde{T}$ with the same properties is then obtained by:

$$\tilde{T}(x, y) := T(x, \bar{y}) \qquad (x, y \in \mathbb{C}^{n+2}),$$

and a real bilinear map is obtained just by replacing $\mathbb{C}$ by $\mathbf{R}$.

We observe that a sesquilinear map $T\colon X \times Y \to Z$ is $(1/n)$-surjective (respectively, $(1/n)$-open), if and only if the map $T^{(n)}\colon X^n \times Y^n \to Z$ is surjective (respectively, open), where

$$T^{(n)}((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \sum_{i=1}^n T(x_i, y_i).$$

Our map $T\colon \mathbb{C}^{n+2} \times \mathbb{C}^{n+2} \to \mathbb{C}^N$ is defined by

$$\begin{aligned} &T((u_1, \dots, u_n, v, w), (x_1, \dots, x_n, y, z)) \\ &\quad = ((u_i \overline{x_j})_{i,j=1}^n, (u_i \bar{y})_{i=1}^n, u_1 \bar{z} + v\overline{x_1} + w\bar{y}, w\overline{x_1} + v\bar{y}). \end{aligned}$$

The reader may verify that in the case $n = 1$ this reduces to Horowitz' example, except for the complex conjugates and order of components. To construct $T^{(n)}$ we simply replace the numbers $u_1, \dots, z \in \mathbb{C}$ by vectors $\mathbf{u}_1, \dots, \mathbf{z} \in \mathbb{C}^n$ and the products $u_i \overline{x_j}$, etc., by scalar products $\mathbf{u}_i \cdot \mathbf{x}_j$, etc.:

$$\begin{aligned} &T^{(n)}((\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}, \mathbf{w}), (\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}, \mathbf{z})) \\ &\quad = ((\mathbf{u}_i \cdot \mathbf{x}_j)_{i,j=1}^n, (\mathbf{u}_i \cdot \mathbf{y})_{i=1}^n, \mathbf{u}_1 \cdot \mathbf{z} + \mathbf{v} \cdot \mathbf{x}_1 + \mathbf{w} \cdot \mathbf{y}, \mathbf{w} \cdot \mathbf{x}_1 + \mathbf{v} \cdot \mathbf{y}). \end{aligned}$$

To show that $T^{(n)}$ is surjective, take any vector

$$\mathbf{h} = ((\xi_{ij})_{i,j=1}^n, (\eta_i)_{i=1}^n, \varsigma, \theta) \in \mathbb{C}^N.$$

If $\eta_i = 0$ for all i, then let

$$\mathbf{x}_j = \mathbf{e}_j \qquad (1 \le j \le n),$$

where $(\mathbf{e}_j)_i = 1$ if $i = j$, 0 if $i \neq j$,

$$\begin{aligned}
\mathbf{y} &= \mathbf{0}, \\
\mathbf{z} &= \mathbf{0}, \\
\mathbf{u}_i &= (\xi_{i1}, \xi_{i2}, \ldots, \xi_{in}) \qquad (1 \leq i \leq n), \\
\mathbf{v} &= (\varsigma, 0, 0, \ldots, 0), \\
\mathbf{w} &= (\theta, 0, 0, \ldots, 0);
\end{aligned}$$

otherwise, let

$$\begin{aligned}
\mathbf{u}_i &= \mathbf{e}_i \qquad (1 \leq i \leq n) \\
\mathbf{x}_j &= (\bar{\xi}_{1j}, \bar{\xi}_{2j}, \ldots, \bar{\xi}_{nj}) \qquad (1 \leq j \leq n), \\
\mathbf{y} &= (\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_n), \\
\mathbf{v} &= \theta(\mathbf{y} \cdot \mathbf{y})^{-1}\mathbf{y}, \\
\mathbf{w} &= \mathbf{0}, \\
\mathbf{z} &= (\varsigma - (\mathbf{v} \cdot \mathbf{x}_1))^{-}\mathbf{u}_1.
\end{aligned}$$

In both cases, $T^{(n)}((\mathbf{u}_1, \ldots, \mathbf{w}), (\mathbf{x}_1, \ldots, \mathbf{z})) = \mathbf{h}$. This proves that $T^{(n)}$ is surjective, so T is $(1/n)$-surjective.

To show that T is not $(1/n)$-open, suppose otherwise. Then, by Remark (c) and the equivalence of norms on finite-dimensional spaces, there exists a number $M \geq 1$ such that, whenever $\mathbf{h} = ((\xi_{ij}), (\eta_i), \varsigma, \theta) \in \mathbb{C}^N$ with $|\xi_{ij}|$, $|\eta_i|$, $|\varsigma|$, $|\theta| \leq 1$ for all i, j, there exist $\mathbf{u}_i, \mathbf{v}, \mathbf{w}, \mathbf{x}_i, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n$ $(1 \leq i \leq n)$ such that

$$T^{(n)}((\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}, \mathbf{w}), (\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}, \mathbf{z})) = \mathbf{h} \tag{2}$$

and

$$\|\mathbf{u}_i\|, \|\mathbf{v}\|, \|\mathbf{w}\|, \|\mathbf{x}_i\|, \|\mathbf{y}\|, \|\mathbf{z}\| \leq M \qquad (1 \leq i \leq n), \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm: $\|\mathbf{x}\| := (\mathbf{x} \cdot \mathbf{x})^{1/2}$ $(\mathbf{x} \in \mathbb{C}^n)$.

Extending Horowitz' argument, we let $\mathbf{h}$ be defined by:

$$\begin{aligned}
\xi_{11} &= a, \qquad \xi_{ii} = 1 \quad (2 \leq i \leq n), \qquad \xi_{ij} = 0 \quad (i \neq j), \\
\eta_1 &= a, \qquad \eta_i = 0 \quad (2 \leq i \leq n), \\
\varsigma &= a, \qquad \theta = 1;
\end{aligned}$$

where $0 < a < 1$ and we shall eventually make a tend to zero. Suppose $\mathbf{u}_i$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}_i$, $\mathbf{y}$, $\mathbf{z}$ satisfy (2) and (3) above. We wish to show that $\|\mathbf{x}_1\|\|\mathbf{u}_1\|$ must be small. This is most conveniently done by the following matrix argument.

Define the $n \times n$ matrices $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$, $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ from the column vectors

$$\begin{aligned}
\mathbf{a}_1 &= a^{-1/2}\mathbf{u}_1, \qquad \mathbf{a}_i = \mathbf{u}_i \quad (2 \leq i \leq n), \\
\mathbf{b}_1 &= a^{-1/2}\mathbf{x}_1, \qquad \mathbf{b}_j = \mathbf{x}_j \quad (2 \leq j \leq n).
\end{aligned}$$

Then the equations $\mathbf{u}_i \cdot \mathbf{x}_j = \xi_{ij}$ $(1 \le i, j \le n)$ become $\mathbf{A}^*\mathbf{B} = \mathbf{I}$. It follows that $\mathbf{A}$ and $\mathbf{B}$ are invertible and $\mathbf{B}\mathbf{A}^* = \mathbf{I} = \mathbf{B}^*\mathbf{A}$, so $\mathbf{B}^*\mathbf{B}\mathbf{A}^*\mathbf{A} = \mathbf{I}$. Hence

$$\begin{aligned} 1 &= (\mathbf{B}^*\mathbf{B}\mathbf{A}^*\mathbf{A})_{11} \\ &= \sum_{i=1}^{n} (\mathbf{b}_i \cdot \mathbf{b}_1)(\mathbf{a}_1 \cdot \mathbf{a}_i) \\ &= a^{-2}\|\mathbf{x}_1\|^2\|\mathbf{u}_1\|^2 + \sum_{i=2}^{n} a^{-1}(\mathbf{x}_i \cdot \mathbf{x}_1)(\mathbf{u}_1 \cdot \mathbf{u}_i). \end{aligned}$$

Since $\|\mathbf{x}_i\|, \|\mathbf{u}_i\| \le M$ $(1 \le i \le n)$, this yields

$$a^{-2}\|\mathbf{x}_1\|^2\|\mathbf{u}_1\|^2 \le 1 + (n-1)a^{-1}\|\mathbf{x}_1\|\|\mathbf{u}_1\|M^2.$$

Hence

$$\|\mathbf{x}_1\|\|\mathbf{u}_1\| \le a(1 + (n-1)M^2) \le anM^2. \tag{4}$$

Next, since $\mathbf{A}$ is invertible, $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ is a basis for $\mathbb{C}^n$. Now $\mathbf{u}_i \cdot (\mathbf{x}_1 - \mathbf{y}) = 0$ for all i, so it follows that $\mathbf{x}_1 = \mathbf{y}$. Then the equation $\mathbf{w} \cdot \mathbf{x}_1 + \mathbf{v} \cdot \mathbf{y} = \theta = 1$ gives

$$(\mathbf{w} + \mathbf{v}) \cdot \mathbf{x}_1 = 1, \tag{5}$$

and so

$$\|\mathbf{w} + \mathbf{v}\|\|\mathbf{x}_1\| \ge 1. \tag{6}$$

We also have

$$\mathbf{u}_1 \cdot \mathbf{z} + \mathbf{v} \cdot \mathbf{x}_1 + \mathbf{w} \cdot \mathbf{y} = \varsigma = a,$$

so

$$\mathbf{u}_1 \cdot \mathbf{z} = a - (\mathbf{v} + \mathbf{w}) \cdot \mathbf{x}_1 = -(1 - a),$$

by (5), and so

$$\|\mathbf{u}_1\| \ge (1 - a)/M. \tag{7}$$

Combining (4) and (7),

$$\|\mathbf{x}_1\| \le anM^3/(1 - a). \tag{8}$$

Finally,

$$\begin{aligned} 2M &\ge \|\mathbf{w} + \mathbf{v}\| \\ &\ge \|\mathbf{x}_1\|^{-1}, && \text{by (6)}, \\ &\ge (1 - a)/anM^3, && \text{by (8)}. \end{aligned}$$

Now M is a constant, independent of a, so letting a tend to zero gives the required contradiction, completing the proof of Theorem 2.

COROLLARY. *For every positive integer n, there is a commutative $(n^2 + n + 3)$-dimensional algebra of operators on the Hilbert space $\mathbb{C}^{2n+5}$ which has property $(\mathbb{A}_{1/n})$, but which, for all $r > 0$, does not have property $(\mathbb{A}_{1/n}(r))$.*

The proof of this is the same as that of [**1**], where Bercovici derives this result for $n = 1$ from Horowitz' example. We therefore omit further discussion and refer the reader to [**1**] for an explanation of the terminology as well as for the proof.

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