

## AN ELEMENTARY PROOF OF TITCHMARSH'S CONVOLUTION THEOREM

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**ABSTRACT.** We give an elementary proof of the following theorem of Titchmarsh. Suppose  $f, g$  are integrable on the interval  $(0, 2T)$  and that the convolution  $f * g(t) = \int_0^t f(t-x)g(x) dx = 0$  on  $(0, 2T)$ . Then there are nonnegative numbers  $\alpha, \beta$  with  $\alpha + \beta \geq 2T$  for which  $f(x) = 0$  for almost all  $x$  in  $(0, \alpha)$  and  $g(x) = 0$  for almost all  $x$  in  $(0, \beta)$ .

Suppose  $f, g$  are integrable on the interval  $(0, 2T)$ . If  $f = 0$  a.e. on  $(0, \alpha)$ ,  $g = 0$  a.e. on  $(0, \beta)$  with  $\alpha + \beta = 2T$ , then the convolution

$$f * g(t) = \int_0^t f(t-x)g(x) dx = 0 \quad \text{for } 0 \leq t \leq 2T.$$

The converse of this statement is the famous Titchmarsh's Convolution Theorem. Let  $f, g$  belong to  $L^1(0, 2T)$  and vanish on  $(-\infty, 0)$ . Suppose that  $\int_0^t f(t-x)g(x) dx = 0$  on  $(0, 2T)$ . Then there are nonnegative numbers  $\alpha, \beta$  with  $\alpha + \beta \geq 2T$  for which  $f(x) = 0$  for almost all  $x$  in  $(0, \alpha)$  and  $g(x) = 0$  for almost all  $x$  in  $(0, \beta)$ .

There are many proofs of this theorem. The first three: Titchmarsh [6], Crum [1] and Dufresnoy [2] were based on the theory of analytic or harmonic functions. An elaborate real variable proof was later given by Mikusinski and Ryll-Nardzewski in [4, (a), (b), (c)]. An entirely different proof, now classical, by the same authors, appears in [5]. Unfortunately it is valid only for  $T = \infty$ . Recently, Helson [3] gave an elegant proof using the theory of Hardy's  $H^p(R)$  spaces and invariant subspaces. The present proof, valid for any  $T$ , is elementary, using no machinery beyond Fubini's theorem and Parseval's formula for trigonometric series.

Put  $F(x) = \int_0^x f(s) ds$ . Then, by Fubini's theorem

$$0 = \int_0^t \int_0^u f(u-x)g(x) dx du = \int_0^t \int_x^t f(u-x)g(x) du dx = \int_0^t F(t-x)g(x) dx.$$

This is  $F * g(t) = 0$ ,  $t \in (0, 2T)$ . Similarly, putting  $G(x) = \int_0^x g(s) ds$  we get  $F * G(t) = 0$ ,  $t \in (0, 2T)$ .

Now if Titchmarsh's theorem is true for the functions  $F, G$ , it is also true for  $f, g$ . Hence, we may assume, in the proof of Titchmarsh's Theorem, that  $f, g$  are several times differentiable and satisfy e.g. relations like  $f' * g = 0$  and  $g(0) = 0$ .

**LEMMA 1.** Suppose that  $h$  is continuous on  $(0, 2T)$  and that

$$(1) \quad \left| \int_0^{2t} e^{2n(t-x)} h(x) dx \right| \leq C_t n^{-1/2}$$

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for  $0 \leq t \leq T$  and all  $n \in \{1, 2, 3, \dots\}$  where  $C_t$  is independent of  $n$ . Then  $h = 0$  on  $(0, T)$ .

PROOF. We have boundedly

$$\lim_{n \rightarrow \infty} [1 - e^{-e^{2n(t-x)}}] = \begin{cases} 1 & \text{for } 0 \leq x < t, \\ 0 & \text{for } t < x \leq 2t. \end{cases}$$

Hence, by dominated convergence

$$(2) \quad \int_0^t h(x) dx = \lim_n \int_0^{2t} [1 - e^{-e^{2n(t-x)}}] h(x) dx.$$

On the other hand, for fixed  $t$  and  $n$ , the sums

$$- \sum_{k=1}^N \frac{(-1)^k}{k!} e^{2nk(t-x)}$$

converge uniformly on  $(0, 2T)$  to  $[1 - e^{-e^{2n(t-x)}}]$  as  $N \rightarrow \infty$ . Hence, by (1)

$$\begin{aligned} \left| \int_0^{2t} [1 - e^{-e^{2n(t-x)}}] h(x) dx \right| &= \lim_N \left| \int_0^{2t} \sum_{k=1}^N \frac{(-1)^k}{k!} e^{2nk(t-x)} h(x) dx \right| \\ &\leq \limsup_N \sum_1^N \frac{1}{k!} C_t (nk)^{-1/2} \leq C_t n^{-1/2} e. \end{aligned}$$

By (2)

$$\int_0^t h(x) dx = 0, \quad t \in (0, T).$$

Hence  $h = 0$  on  $(0, T)$ .

LEMMA 2. If  $f * g(t) = 0$  for  $t \in (0, 2T)$  then

$$f(x)g(x) = 0 \quad \text{for all } x \in (0, T).$$

PROOF. For  $t \leq T$  we calculate in two ways the integral

$$I_t = \iint_{\Delta \cup \Delta'} e^{n(2t-y)} e^{-iky} f(y-x)g(x) dx dy,$$

where  $\Delta$  is the triangle  $0 \leq x \leq y, 0 \leq y \leq 2t$  and  $\Delta'$  is the triangle  $2t \leq y \leq 4t, y - 2t \leq x \leq 2t$ . Thus  $\Delta \cup \Delta'$  is the parallelogram  $x \leq y \leq 2t + x, 0 \leq x \leq 2t$ . We have

$$I_t = \int_0^{2t} \left\{ \int_x^{2t+x} e^{n(2t-y)} e^{-iky} f(y-x) dy \right\} g(x) dx.$$

Putting  $y = x + u$  the inner integral becomes

$$\int_0^{2t} e^{n(t-u)} e^{-iku} f(u) du e^{n(t-x)} e^{-ikx}.$$

Hence

$$(3) \quad I_t = \int_0^{2t} e^{n(t-u)} e^{-iku} f(u) du \cdot \int_0^{2t} e^{n(t-x)} e^{-ikx} g(x) dx.$$

On the other hand, the integral over  $\Delta$  is zero by  $\int_0^y f(y-x)g(x) dx = 0$ . Hence

$$(4) \quad I_t = \iint_{\Delta'} = \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h_t(y) dy$$

where

$$h_t(y) = \int_{y-2t}^{2t} f(y-x)g(x) dx.$$

We have

$$h_t(2t) = \int_0^{2t} f(2t-x)g(x) dx = 0, \quad h_t(4t) = 0.$$

An integration by parts in (4) then gives

$$(4') \quad I_t = (n+ik)^{-1} \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h'_t(y) dy.$$

We have

$$h'_t(y) = -f(2t)g(y-2t) + \int_{y-2t}^{2t} f'(y-x)g(x) dx, \\ h'_t(2t) = 0, \quad h'_t(4t) = -f(2t)g(2t).$$

Hence, by (4')

$$(4'') \quad I_t = (n+ik)^{-2} e^{-2tn} e^{-i4tk} f(2t)g(2t) \\ + (n+ik)^{-2} \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h''_t(y) dy.$$

In the last integral the coefficient of  $h''_t(y)$  has a modulus bounded by 1 on the interval of integration and we may suppose that  $|f(x)g(x)| \leq M$  for  $x$  in  $(0, 2T)$ ,  $|h''_t(y)| \leq M$  for  $t \in (0, T)$ ,  $y \in (2t, 4t)$  where  $M$  is a constant.

Hence, by (4''),  $|I_t| \leq |n+ik|^{-2}(M+2tM) \leq |n+ik|^{-2}C$ , where  $C$  is a constant. By (3)

$$(5) \quad \left| \int_0^{2t} e^{n(t-x)} e^{-ikx} f(x) dx \right| \cdot \left| \int_0^{2t} e^{n(t-x)} e^{-ikx} g(x) dx \right| \leq |n+ik|^{-2}C.$$

We now use the polarized version of Parseval's formula, viz.

$$\left| \frac{1}{2\pi} \int_0^{2t} f\bar{g} \right| = \left| \sum_p \hat{f}(p)\overline{\hat{g}(p)} \right| \leq \sum_p |\hat{f}(p)\hat{g}(p)|.$$

Putting  $k = k_p = 2\pi(2t)^{-1}p$ , we get from (5)

$$\left| \frac{1}{2\pi} \int_0^{2t} e^{2n(t-x)} f(x)g(x) dx \right| \leq \sum_{p=-\infty}^{\infty} |n+ik_p|^{-2}C \\ \leq n^{-1/2} \sum_p |n+ik_p|^{-3/2}C \leq n^{-1/2}C_t,$$

where  $C_t$  is independent of  $n$ . By Lemma 1,  $fg = 0$  on  $(0, T)$ .

**PROOF OF TITCHMARSH'S THEOREM.** Observe first that if  $\gamma > 0$  then the translate  $f_\gamma(x) = f(x-\gamma)$  satisfies the equation  $f_\gamma * g(t) = 0$  for  $t \in (0, 2T)$ .

In the other direction let  $(0, \alpha)$  and  $(0, \beta)$  respectively be the largest intervals on which  $f$  and  $g$  vanish.

We have

$$\int_0^t f_{-\alpha}(t-x)g(x) dx = \int_0^t f(t+\alpha-x)g(x) dx = \int_0^{t+\alpha} f(t+\alpha-x)g(x) dx = 0$$

for  $t+\alpha \in (0, 2T)$ , hence for  $t \in (0, 2T-\alpha)$ . We infer that

$$f_{-\alpha} * g_{-\beta}(t) = 0 \quad \text{for } t \in (0, 2T-\alpha-\beta).$$

By the observation above we have

$$f_{-\alpha+\gamma} * g_{-\beta}(t) = 0 \quad \text{for } t \in (0, 2T-\alpha-\beta) \text{ and all } \gamma > 0,$$

and by Lemma 2

$$(6) \quad f_{-\alpha+\gamma}(x)g_{-\beta}(x) = 0 \quad \text{for } x \in \left(0, T - \frac{\alpha+\beta}{2}\right) \text{ and all } \gamma > 0.$$

Assume that  $2T-\alpha-\beta > 0$ ; then  $g_{-\beta}(x_0) \neq 0$  for some  $x_0 \in (0, T - (\alpha+\beta)/2)$ . If  $0 < \gamma < x_0$  then, by (6), applied to  $x_0$ , we get  $f_{-\alpha+\gamma}(x_0) = 0$ , i.e.  $f_{-\alpha}(x_0-\gamma) = 0$ . This last is  $f_{-\alpha}(u) = 0$  for  $u \in (0, x_0)$  which is impossible. Thus  $2T - (\alpha + \beta) \leq 0$  and the theorem is proved.

ADDED IN PROOF. J. G. Mikusinski, in his book *The Bochner integral*, Academic Press, New York, 1978, shows that the proof in [5] can be made valid for any  $T > 0$ .

#### BIBLIOGRAPHY

1. M. M. Crum, *On the resultant of two functions*, Quart. J. Math. Oxford Ser. 12, no. 46, 1941, pp. 108-111.
2. J. Dufresnoy, *Sur le produit de composition de deux fonctions*, C. R. Acad. Sci. Paris 335 (1947), 857-859.
3. H. Helson, *Harmonic analysis*, Addison-Wesley, Reading, Mass., 1983.
4. (a) J. G. Mikusinski, *On generalized exponential functions*, Studia Math. 13 (1951), 48-50.  
(b) J. G. Mikusinski and C. Ryll-Nardzewski, *A theorem on bounded moments*, Studia Math. 13 (1952), 51-55.  
(c) J. G. Mikusinski, *A new proof of Titchmarsh's convolution theorem*, Studia Math. 13 (1952), 56-58.
5. ———, *Operational calculus*, Pergamon Press, 1959.
6. E. C. Titchmarsh, *The zeros of certain integral functions*, Proc. London Math. Soc. (2) 25 (1926), 283-302.

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