PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 1, September 1988

INEQUALITIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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(Communicated by William J. Davis)

ABSTRACT. A moment inequality is proved for sums of independent random variables in the Lorentz spaces $L_{p,q}$, thus extending an inequality of Rosenthal. The latter result is used in combination with a square function inequality to give a proof of a Banach space isomorphism theorem. Further moment inequalities are also proved.

1. Introduction. First some notation and standard terminology. Let (M, Σ, m) be a measure space. For $1 < p < \infty$ and $0 < q \leq \infty$ the Lorentz space $L_{p,q}(M)$ is the collection of all measurable f on M such that

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty f^*(t)^q t^{q/p-1}\,dt\right)^{1/q} < \infty \qquad (0 < q < \infty)$$
$$\|f\|_{p,\infty} = \operatorname*{ess\,sup}_{t>0} t^{1/p} f^*(t) < \infty,$$

where f^* denotes the decreasing rearrangement of $|f|$. Note that $L_{p,p}(M)$ is just the Lebesgue space $L_p(M)$ and that $L_{p,\infty}$ is the space weak-L_p. The $L_{p,q}$ spaces arise in the Lions-Peetre K-method of interpolation: in particular,

$$L_{p,q} = [L_{p_1}, L_{p_2}]_{\theta,q}, \quad \text{where } 1/p = (1-\theta)/p_1 + \theta/p_2.$$

Standard facts about $L_{p,q}$ and the K-method will be used throughout; these may be found in, for example, [**13** and **1**].

To avoid awkward formulae we reserve the letter C to denote a positive constant (whose precise value may change from line to line) depending only on the parameters p, q, etc. We write $A \sim B$ when there exists such a constant C with $(1/C)A \leq B \leq CA$.

In [**11**] Rosenthal proved the following moment inequality for independent symmetric random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

Received by the editors June 1, 1987 and, in revised form, September 15, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46E30, 60G50; Secondary 46B25, 60B11.

Key words and phrases. Lorentz function spaces, sums of independent random variables, Lions-Peetre interpolation.

The second author was supported in part by NSF DMS-8500764.

THEOREM. *Given $2 < p < \infty$ there exists a constant C, depending only on p, such that*

(1)

$$\max\left(\left\|\sum_{k=1}^n X_k\right\|_2, \left(\sum_{k=1}^n \|X_k\|_p^p\right)^{1/p}\right) \le \left\|\sum_{k=1}^n X_k\right\|_p$$
$$\le C\max\left(\left\|\sum_{k=1}^n X_k\right\|_2, \left(\sum_{k=1}^n \|X_k\|_p^p\right)^{1/p}\right)$$

for all independent symmetric random variables $X_1, X_2, \ldots, X_n$ in $L_p(\Omega)$.

In this paper we give some extensions and applications of Rosenthal's inequality. In §2 we prove an analogue for the $L_{p,q}$ norm of a sum of independent symmetric random variables. In §3 we show how (1) can be used in conjunction with a square function inequality of Stein to give a short proof of an isomorphism result from [**8**]. The paper concludes with some further consequences of Stein's inequality.

Finally, we should like to thank Bill Johnson and Joel Zinn for several enlightening conversations about Rosenthal's inequality, and Gideon Schechtman for showing us the short proof of Proposition 3.1.

2. Extension to $L_{p,q}$. Given a measurable function f on a measure space (M, Σ, m) define its distribution d_f by $d_f(t) = m(\{|f| \ge t\})$, and given random variables $f_1, f_2, \ldots, f_n$ on the probability space Ω define the disjoint sum, denoted $\sum_{k=1}^n \bigoplus f_k$, to be any function f on $(0, \infty)$ with $d_f(t) = \sum_{k=1}^n d_{f_k}(t)$. Recall also that the intersection of two Lorentz spaces $L_{p_1,q_1}(0,\infty)$ and $L_{p_2,q_2}(0,\infty)$ is a quasi-Banach space under the quasi-norm $\max(\|f\|_{p_1,q_1}, \|f\|_{p_2,q_2})$.

LEMMA 2.1. *Let $0 < r < s < \infty$, $0 < \theta < 1$, and $0 < q \le \infty$. Then*

$$[L_r, L_r \cap L_s]_{\theta,q} = L_r \cap [L_r, L_s]_{\theta,q},$$

where L_p denotes $L_p(0,\infty)$ throughout.

PROOF. It is easily seen that $[L_r, L_r \cap L_s]_{\theta,q} \subset L_r \cap [L_r, L_s]_{\theta,q}$. To show the reverse inclusion fix $f \in L_r \cap [L_r, L_s]_{\theta,q}$ and let $K(t) = K(t, f; L_r, L_r \cap L_s)$. Then for $t \ge 1$, $K(t) \le \|f\|_r = K(t, f; L_r, L_r)$. For $t < 1$ we use Holmstedt's formula [**7**]: set $1/\alpha = 1/r - 1/s$, then

$$\begin{aligned} K(t) &\le \|f^* I(0, t^\alpha)\|_r + t\|f^* I(t^\alpha, \infty)\|_{L_r \cap L_s} \\ &\le \|f^* I(0, t^\alpha)\|_r + t\|f^* I(t^\alpha, \infty)\|_s + t\|f\|_r \\ &\sim K(t, f; L_r, L_s) + K(t, f; L_r, L_r). \end{aligned}$$

The desired inclusion now follows easily.

THEOREM 2.2. *Given $2 < p < \infty$ and $0 < q \le \infty$, there exists a constant C (depending only on p and q) such that*

(2)

$$\frac{1}{C}\max\left(\left\|\sum_{k=1}^{n} X_k\right\|_2, \left\|\sum_{k=1}^{n}\bigoplus X_k\right\|_{p,q}\right) \le \left\|\sum_{k=1}^{n} X_k\right\|_{p,q} \le C\max\left(\left\|\sum_{k=1}^{n} X_k\right\|_2, \left\|\sum_{k=1}^{n}\bigoplus X_k\right\|_{p,q}\right)$$

for all independent symmetric random variables $X_1, X_2, \dots, X_n$ in $L_{p,q}(\Omega)$.

PROOF. It is convenient to take Ω to be $[0,1]^{\mathbf{N}}$ with the product measure and to denote a typical element of Ω by the sequence $s = (s_1, s_2, \dots)$. First observe that the theorem simply reduces to Rosenthal's inequality in the case $p = q$. Define a linear operator $T\colon L_0(0,\infty) \to L_0(\Omega \times [0,1])$ by $T(f) = \sum_{k=1}^{\infty} f_k(s_k) r_k(t)$, where $f_k(s_k) = f(k-1+s_k)$ and $r_k(t)$ is the kth Rademacher function. Then by (1) T is a bounded operator (in fact an isomorphic embedding) from $L_2(0,\infty) \cap L_p(0,\infty)$ into $L_p(\Omega \times [0,1])$ for $p > 2$. So by Lemma 2.1 and Marcinkiewicz interpolation T is bounded from $L_2(0,\infty) \cap L_{p,q}(0,\infty)$ into $L_{p,q}(\Omega \times [0,1])$, which proves the right-hand inequality of (2). Now

$$\left\|\sum_{k=1}^{n} X_k\right\|_2 \le C\left\|\sum_{k=1}^{n} X_k\right\|_{p,q}$$

since $p > 2$, and so to prove the left-hand inequality it is enough to show that

$$\left\|\sum_{k=1}^{n} \oplus X_k\right\|_{p,q} \le C\left\|\sum_{k=1}^{n} X_k\right\|_{p,q}.$$

Since $\sum_{k=1}^{n} X_k$ has the same distribution as $\sum_{k=1}^{n} X_k(s) r_k(t)$, by the generalization of Khintchine's inequality to lattices [**10**, Theorems 1.d.6 and 2.b.11] it is enough to show that

$$\left\|\sum_{k=1}^{n} \oplus X_k\right\|_{p,q} \le C\left\|\left(\sum_{k=1}^{n} X_k^2\right)^{1/2}\right\|_{p,q},$$

and this follows from [**5**, Corollary 2.7] for $p > 2$.

REMARK 2.3. Further inequalities for sums of random variables in $L_{p,q}$ are proved in [**4**]. These are used in [**5**] to study the subspace structure of $L_{p,q}(0,\infty)$. The problem of interpolating between arbitrary intersections of L_p spaces along the lines of Lemma 2.1 is solved in [**6**].

3. Further results. Let Y_p denote the space $L_p(0,\infty) \cap L_2(0,\infty)$ for $2 \le p \le \infty$ and $L_p(0,\infty) + L_2(0,\infty)$ for $0 < p \le 2$. Note that

$$\|f\|_{Y_p} \sim \|f^* I(0,1)\|_p + \|f^* I(1,\infty)\|_2$$

and that $(Y_p)^* = Y_q$ if $p \ge 1$ and if $1/p + 1/q = 1$. Using the operator T of Theorem 2.2 we give a proof of the beautiful theorem from [**8**] which states that Y_p and $L_p(0,1)$ are isomorphic Banach spaces for $1 < p < \infty$. This is a rather different approach from that of [**8**], in which a stochastic integral with respect to a symmetrized Poisson process is used to embed Y_p into L_p. We need the following square function inequality, which follows easily from a result of Stein. It is also implicit in a result of Bryc and Kwapień [**2**].

PROPOSITION 3.1. *Let $1 < p < \infty$ and let $(\mathcal{F}_k)_{k=1}^n$ be independent sub-σ-fields of $\mathcal{F}$. There exists C_p such that*

$$\left\| \left(\sum_{k=1}^{n} (E(f_k | \mathcal{F}_k))^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{k=1}^{n} f_k^2 \right)^{1/2} \right\|_p$$

for all $n \geq 1$ and for all $f_1, f_2, \ldots, f_n$ in $L_p(\Omega)$.

PROOF. This is a theorem of Stein [**12**] when $(\mathcal{F}_k)_{k=1}^n$ is replaced by an increasing sequence $(\mathcal{G}_k)_{k=1}^n$ of sub-σ-fields. Let $\mathcal{G}_k$ and $\widetilde{\mathcal{G}}_k$ be the sub-σ-fields generated by $\bigcup_{i=1}^k \mathcal{F}_i$ and $\bigcup_{i=n-k}^n \mathcal{F}_i$, respectively. Applying Stein's square function inequality first with respect to $(\mathcal{G}_k)_{k=1}^n$ and then with respect to $(\widetilde{\mathcal{G}}_k)_{k=1}^n$ gives the result.

THEOREM 3.2. *Y_p and $L_p(0,1)$ are isomorphic for $1 < p < \infty$.*

PROOF. Take Ω to be $[0,1]^{\mathbf{N}}$ as in Theorem 2.2 and identify $\mathcal{F}$ with the sub-σ-field $\{F \times [0,1] \colon F \in \mathcal{F}\}$ of the product σ-field in $\Omega \times [0,1]$. It was observed in the proof of Theorem 2.2 that the operator $T \colon Y_p \to L_p(\Omega \times [0,1])$ is an isomorphic embedding for $2 \leq p < \infty$. Define $P \colon L_p(\Omega \times [0,1]) \to L_p(\Omega \times [0,1])$ by

$$P(f) = \sum_{k=1}^{\infty} E(f r_k | s_k) r_k(t)$$

(recall that $s_1, s_2, \ldots$ are the independent coordinates of an element s in Ω). Then P is a selfadjoint projection onto the range of T. To see that P is bounded for $1 < p < \infty$, note that by Khintchine's inequality and Proposition 3.1 we have

$$\begin{aligned} \|Pf\|_p &\sim \left\| \left(\sum_{k=1}^{\infty} (E(f r_k | s_k))^2 \right)^{1/2} \right\|_p \\ &\leq C_p \left\| \left(\sum_{k=1}^{\infty} (E(f r_k | \mathcal{F}))^2 \right)^{1/2} \right\|_p \\ &\sim \left\| \sum_{k=1}^{\infty} E(f r_k | \mathcal{F}) r_k \right\|_p . \end{aligned}$$

But

$$\left\| \sum_{k=1}^{\infty} E(f r_k | \mathcal{F}) r_k \right\|_p \leq C_p \|f\|_p$$

for $1 < p < \infty$: this is just the conditional form of the well-known fact that the Rademacher sequence spans a complemented subspace of L_p. Since P is selfadjoint it follows that P is a bounded projection onto Y_p (i.e. the range of T) for all

$1 < p < \infty$. Since $L_p(0,1)$ is plainly isomorphic to a complemented subspace of Y_p, Pełczýnski's decomposition argument (e.g. [**10**]) will complete the proof.

The existence of the projection P in the last result implies the following extension of Rosenthal's inequality to the range $1 < p < 2$; this extension was first proved by Johnson and Schechtman [**9**].

COROLLARY 3.3. *Let* $1 < p < 2$. *Then*

$$\left\| \sum_{k=1}^{n} X_k \right\|_p \sim \left\| \sum_{k=1}^{n} \bigoplus X_k \right\|_{Y_p}$$

for all independent symmetric random variables $X_1, X_2, \ldots, X_n$.

A different proof of this result will appear in [**9**] together with its generalization to the setting of rearrangement invariant spaces.

We conclude with some further consequences of Proposition 3.1.

COROLLARY 3.4. *Let* $1 < p \leq \infty$ *and let* $(\mathcal{F}_k)_{k=1}^n$ *be independent sub-σ-fields of* $\mathcal{F}$. *Then*

$$\left\| \left(\sum_{k=1}^{n} (E(f|\mathcal{F}_k))^2 \right)^{1/2} \right\|_p \leq C_p \sqrt{n}\, \|f\|_p$$

for all $f \in L_p(\Omega)$.

PROOF. Take $f_k = f$ $(1 \leq k \leq n)$ in Proposition 3.1 in the range $1 < p < \infty$. The case $p = \infty$ is obvious because the conditional expectation operator is a contraction on $L_\infty(\Omega)$.

REMARK 3.5. Observe that the choice of $f \equiv 1$ shows that the constant $\sqrt{n}$ is of the correct order. Corollary 3.4 is only interesting in the range $1 < p < 2$, for in the range $2 \leq p < \infty$ it follows easily from Minkowski's inequality. The result breaks down completely when $p = 1$. To see this, let $X_1, X_2, \ldots, X_n$ be independent nonnegative random variables with $EX_k = 1$. Then $E(\prod_{i=1}^n X_i | X_k) = X_k$ and $\| \prod_{k=1}^n X_k \|_1 = 1$. Taking $X_k = |\theta_k|$, where $\theta_1, \theta_2, \ldots, \theta_n$ are independent symmetric r-stable random variables $(1 < r < 2)$, gives $\|(\sum_{k=1}^n X_k^2)^{1/2}\|_1 \sim n^{1/r}$. So when $p = 1$ the factor of $\sqrt{n}$ in Corollary 3.4 must be replaced by n.

Our final theorem should be viewed as a generalization of Khintchine's inequality.

THEOREM 3.6. *Let* $1 < p < 2$ *and let* $X_1, X_2, \ldots, X_n$ *be independent symmetric random variables. Then*

$$\frac{\| \sum_{k=1}^n X_k \|_p}{\prod_{k=1}^n \|X_k\|_p} \leq \frac{C_p (\sum_{k=1}^n \|X_k\|_1^2)^{1/2}}{\prod_{k=1}^n \|X_k\|_1}$$

PROOF. Let $a_1, a_2, \ldots, a_n$ be real numbers. Applying Proposition 3.1 with $f_k = a_k f$ $(1 \leq k \leq n)$ gives

$$\left\| \left(\sum_{k=1}^{n} (a_k E(f|\mathcal{F}_k))^2 \right)^{1/2} \right\|_p \leq C_p \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \|f\|_p.$$

Let

$$X = \prod_{k=1}^{n} \frac{|X_k|}{\|X_k\|_1}.$$

Then

$$\|X\|_p = \prod_{k=1}^{n} \frac{\|X_k\|_p}{\|X_k\|_1}$$

and

$$E(X|X_k) = \frac{|X_k|}{\|X_k\|_1}.$$

Putting $f = X$ and $a_k = \|X_k\|_1$ gives

$$\left\| \left(\sum_{k=1}^{n} X_k^2 \right)^{1/2} \right\|_p \le C_p \left(\sum_{k=1}^{n} \|X_k\|_1^2 \right)^{1/2} \frac{\prod_{k=1}^{n} \|X_k\|_p}{\prod_{k=1}^{n} \|X_k\|_1},$$

while by Khintchine's inequality

$$\left\| \sum_{k=1}^{n} X_k \right\|_p \sim \left\| \left(\sum_{k=1}^{n} X_k^2 \right)^{1/2} \right\|_p .$$

The desired conclusion now follows.

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