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THE DERIVATIVE OF BAZILEVIČ FUNCTIONS

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ABSTRACT. For $\alpha > 0$, let $B_1(\alpha)$ be the class of normalised analytic functions f defined in the open unit disc D such that $\operatorname{Re}(f(z)/z)^{\alpha-1}f'(z) > 0$ for $z \in D$. Sharp upper and lower bounds are obtained for $|zf'(z)/f(z)|$ when $f \in B_1(\alpha)$.

1. Introduction. For $\alpha > 0$, denote by $B(\alpha)$ the class of analytic Bazilevic functions defined in the unit disc D, with $f(0) = 0$ and $f'(0) = 1$ (e.g. [**2**, **8**]) and by $B_1(\alpha)$ the subclass of $B(\alpha)$ for which

$$\operatorname{Re} f'(z)(f(z)/z)^{\alpha-1} > 0 \tag{1}$$

for $z \in D$ [**7**]. Clearly $B_1(1) = R$, the class of analytic functions satisfying $\operatorname{Re} f'(z) > 0$ in D first studied by Alexander [**1**].

In [**9**], it was shown that for $f \in R$ and $z \in D$,

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{-K}{(1-|z|)\log(1-|z|)},$$

where K is an absolute constant. Recently, London [**5**] obtained the sharp upper bound and Gray and Ruscheweyh [**4**], the sharp upper and lower bounds for $|zf'(z)/f(z)|$ when $f \in R$.

In this paper, we give sharp upper and lower bounds for the wider class $B_1(\alpha)$. This sharpens the upper bound estimate given by El-Ashwah and Thomas [**3**].

2. Results. Following Gray and Ruscheweyh (loc. cit), we begin by defining a slightly wider class of functions.

DEFINITION. For $\alpha > 0$, denote by $B_0(\alpha)$ the class of function analytic in D with $f(0) = 0$, $f'(0) = 1$ and satisfying the condition

$$\operatorname{Re} e^{i\phi} f'(z)(f(z)/z)^{\alpha-1} > 0$$

for $z \in D$ and for some $\phi = \phi(f) \in \mathbf{R}$.

THEOREM. *For $f \in B_0(\alpha)$ and $|z| \le r < 1$,*

$$\frac{1-r}{\alpha(1+r)}\int_0^1 t^{\alpha-1}\frac{1-tr}{1+tr}\,dt \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+r}{\alpha(1-r)}\int_0^1 t^{\alpha-1}\frac{1+tr}{1-tr}\,dt.$$

The left-hand and right-hand inequalities are sharp in $B_0(\alpha)$ for the function

$$f_0(z) = z\left(\alpha\int_0^1 t^{\alpha-1}\frac{1+tz}{1-tz}\,dt\right)^{1/\alpha}$$

at $z = -r$ and $z = r$ respectively.

We use the method of Gray and Ruscheweyh (loc. cit) and require the following lemma.

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LEMMA. *Let* $F(z) = 1 - z^\alpha/(\alpha \int_0^z \varsigma^{\alpha-1}/(1-\varsigma)\,d\varsigma)$ *and* $G(z) = (1-F(z))/(1-z)$. *Then* F *and* G *have nonnegative Taylor coefficients about* $z = 0$, *and in particular for* $|z| \le r < 1$,

$$|F(z)| \le F(r) < \lim_{t\to 1} F(t) = 1, \tag{2}$$

$$|F'(z)| \le F'(r) \tag{3}$$

and

$$|G(z)| \le G(r). \tag{4}$$

PROOF. Let

$$H(z) = F(z) - 1 = -z^\alpha / \left(\alpha \int_0^z \frac{\varsigma^{\alpha-1}}{1-\varsigma}\,d\varsigma\right).$$

Then clearly

$$(1-z)(zH'(z) - \alpha H(z)) = \alpha H^2(z). \tag{5}$$

With $H(z) = \sum_{k=0}^\infty c_k z^k$, (5) implies that

$$(k-\alpha)c_k = (k-1-\alpha)c_{k-1} + \alpha \sum_{j=0}^{k} c_j c_{k-j},$$

where $c_{-1} = 0$. Thus

$$c_0 = -1, \qquad c_1 = \frac{\alpha}{\alpha+1}, \qquad c_2 = \frac{\alpha}{(2+\alpha)(\alpha+1)^2} \tag{6}$$

and for $k \ge 3$,

$$(k+\alpha)c_k = \left(k + \frac{\alpha^2 - 2\alpha - 1}{\alpha+1}\right) c_{k-1} + b_k, \tag{7}$$

where

$$b_3 = 0 \quad \text{and} \quad b_k = \alpha \sum_{j=2}^{k-2} c_j c_{k-j} \quad \text{for } k \ge 4.$$

Since $3 + (\alpha^2 - 2\alpha - 1)/(\alpha+1) > 0$ a simple induction argument using (6) and (7) shows that $c_k > 0$ for $k \ge 1$. Thus the coefficients of F are nonnegative and (2) and (3) follow. Finally, with $G(z) = \sum_{k=0}^\infty d_k z^k$, we have

$$d_k = 1 - \sum_{j=1}^{k} c_j = 1 - \lim_{t\to 1} \sum_{j=1}^{k} c_j t^j \ge 1 - \lim_{t\to 1} F(t) = 0$$

and (4) follows.

PROOF OF THE THEOREM. From (1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{h(z)}{\alpha z^{-\alpha} \int_0^z \varsigma^{\alpha-1} h(\varsigma)\,d\varsigma} = \frac{h(z)}{\alpha \int_0^1 t^{\alpha-1} h(tz)\,dt} \tag{8}$$

where $\operatorname{Re} e^{i\phi} h(z) > 0$ for $z \in D$. It follows from the Duality Principle [**6**, Theorem 1.1, Corollary 1.1 and Theorem 1.6] that any value assumed by the right-hand side of (8) for some $z \in D$ is also assumed for this z when h is a function of the form

$(1+xz)/(1+yz)$ where $|x|$, $|y|=1$. Clearly in obtaining upper and lower bounds for $|zf'(z)/f(z)|$, we may take

$$h(z)=\frac{1+xz}{1-z}\quad \text{for } |x|=1. \tag{9}$$

We first obtain the lower bound in the Theorem. Using (8) and (9), we write

$$\begin{aligned}\frac{f(z)}{zf'(z)}&=\frac{\alpha}{z^\alpha}\frac{1-z}{1+xz}\int_0^z \varsigma^{\alpha-1}\frac{1+x\varsigma}{1-\varsigma}\,d\varsigma\\&=\alpha\int_0^1 t^{\alpha-1}\frac{1-z}{1+xz}\cdot\frac{1+xtz}{1-tz}\,dt.\end{aligned}$$

Now for $0\le t\le 1$ and $|z|<1$,

$$\frac{1+t|z|}{1+|z|}\le\left|\frac{1+tz}{1+z}\right|\le\frac{1-t|z|}{1-|z|}.$$

Thus

$$\left|\frac{1+xtz}{1+xz}\,\frac{1-z}{1-tz}\right|\le\frac{1-t|z|}{1-|z|}\,\frac{1+|z|}{1+t|z|}$$

and so

$$\left|\frac{f(z)}{zf'(z)}\right|\le\alpha\frac{1+r}{1-r}\int_0^1 t^{\alpha-1}\frac{1-tr}{1+tr}\,dt$$

which is the required lower bound.

For the upper bound, we use (9) together with F as defined in the Lemma to write

$$\begin{aligned}\alpha\int_0^z\varsigma^{\alpha-1}h(\varsigma)\,d\varsigma&=\alpha\int_0^z\varsigma^{\alpha-1}\left(-x+\frac{x+1}{1-\varsigma}\right)d\varsigma\\&=z^\alpha\frac{1+xF(z)}{1-F(z)}.\end{aligned}$$

Hence (8) and (9) give

$$\frac{zf'(z)}{f(z)}=G(z)\frac{1+xz}{1+xF(z)},$$

where $G(z)=(1-F(z))/(1-z)$. Since $(1+az)/(1+bz)$ maps the closed unit disc onto the circle centre $(1-a\bar{b})/(1-|b|^2)$, radius $|a-b|/(1-|b|^2)$ provided $|b|<1$, we deduce that

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}\right|&\le|G(z)|\frac{|z-F(z)|+|1-F(z)\bar{z}|}{1-|F(z)|^2}\\&=\frac{|G(z)|}{1-|F(z)|^2}\left(r\left|1-\frac{F(z)}{z}\right|+\left|1-r^2+r^2\left(1-\frac{F(z)}{z}\right)\right|\right)\\&\le\frac{|G(z)|}{1-|F(z)|^2}\left(r(1+r)\left|1-\frac{F(z)}{z}\right|+(1-r^2)\right)\\&=\frac{1+r}{1-|F(z)|^2}\left(\frac{r}{\alpha}|F'(z)|+(1-r)|G(z)|\right)\end{aligned}$$

where we have used $F'(z)=\alpha G(z)(1-F(z)/z)$.

It now follows from the Lemma that the last expression is maximal for $z = r$ and so

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{(1+r)G(r)}{1+F(r)} = \frac{1+r}{1-r}\,\frac{1-F(r)}{1+F(r)}$$
$$= \frac{1+r}{1-r}\left(-1+2\alpha r^{-\alpha}\int_0^r \frac{\varsigma^{\alpha-1}}{1-\varsigma}\,d\varsigma\right)^{-1}$$
$$= (1+r)\left(\alpha(1-r)\int_0^1 t^{\alpha-1}\frac{1+tr}{1-tr}\,dt\right)^{-1}$$

which completes the proof.

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