EXISTENCE OF AD-NILPOTENT ELEMENTS AND SIMPLE LIE ALGEBRAS WITH SUBALGEBRAS OF CODIMENSION ONE

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ABSTRACT. For a perfect field $F$ of arbitrary characteristic, the following statements are proved to be equivalent: (1) Any Lie algebra over $F$ contains an ad-nilpotent element. (2) There are no simple Lie algebras over $F$ having only abelian subalgebras. They are used to guarantee the existence of an ad-nilpotent element in every Lie algebra over a perfect field of type $C_1$ of arbitrary characteristic (in particular, over any finite field). Furthermore, we give a sufficient condition to insure the existence of ad-nilpotent elements in a Lie algebra over any perfect field. As a consequence of this result we obtain an easy proof of the fact that the Zassenhaus algebras and $sl(2, F)$ are the only simple Lie algebras which have subalgebras of codimension 1, whenever the ground field $F$ is perfect with char($F$) $\neq 2$. All Lie algebras considered are finite dimensional.

Introduction. It is well known that every Lie algebra over an algebraically closed field contains ad-nilpotent elements. In characteristic zero, it follows from the classical theory. Benkart and Isaacs gave in [3] a proof which works for algebraically closed fields of arbitrary characteristic.

In this paper, we prove that for a nonzero nilpotent derivation $D$ of a Lie algebra $L$, each element of Ker $D$ which acts nilpotently on Ker $D$ is ad-nilpotent on $L$. We use this result to prove that for a perfect field $F$ the following statements are equivalent: (1) Any Lie algebra over $F$ contains an ad-nilpotent element. (2) There are no simple Lie algebras over $F$ having only abelian subalgebras. Then we show that the following holds: (1) Any Lie algebra over a perfect field of type $C_1$ of arbitrary characteristic necessarily contains an ad-nilpotent element. (2) The Brauer group of a perfect field over which every Lie algebra contains ad-nilpotent elements must be trivial (it follows from [6]).

Obviously, Lie algebras containing no nonzero element $x$ such that ad $x$ is split cannot contain any ad-nilpotent element. The simplest example of these algebras is the 3-dimensional Lie algebra $su(2)$ of the real vectors with the vector product.

We prove that the existence of an element $x$ in each subalgebra $S$ of $L$ such that ad $x$ is split on $S$ guarantees the existence of an ad-nilpotent element in $L$, provided $F$ is perfect of arbitrary characteristic. Theorem 1.5 below is slightly stronger than this assertion which is obtained from it by taking the subspace $V$ in the statement of the theorem to be zero.
In §2, we use Theorem 1.5 to study Lie algebras \( L \) with a subalgebra of codimension 1 which contains no nonzero ideal of \( L \). These algebras have been studied by several authors, among them Hofmann [8] who showed that such a simple Lie algebra \( L \) over the real number field \( \mathbb{R} \) must be isomorphic to \( \text{sl}(2, \mathbb{R}) \). Amayo in [1] asserted that over any field \( F \) of characteristic \( p > 2 \), \( L \) is either 2-dimensional, isomorphic to \( \text{sl}(2, F) \) or a Zassenhaus algebra. However, Benkart, Isaacs and Osborn gave in [4] counterexamples which show that Amayo’s result is not true when \( F \) is not perfect. Their results on self-centralizing ad-nilpotent elements yield easy proofs of two special cases of Amayo’s assertion: when \( F \) is algebraically closed with \( \text{char}(F) \neq 2 \) and when \( L \) contains a self-centralizing ad-nilpotent element provided \( \text{char}(F) \neq 2 \) (see [4, p. 284]).

By using our Theorem 1.5, we quickly prove that a Lie algebra \( L \) as in the preceding paragraph necessarily contains a self-centralizing ad-nilpotent element, whenever \( F \) is perfect of arbitrary characteristic. Then, it follows that Amayo’s assertion is true when \( F \) is perfect with \( \text{char}(F) \neq 2 \). For simple Lie algebras, this result was also proved in [5] with techniques quite different.

1. On the existence of ad-nilpotent elements. Let \( L \) be a finite dimensional Lie algebra over an arbitrary field \( F \). For \( x \in L – (0) \), we will denote by \( E_L(x) \) the Engel subalgebra of \( L \) relative to \( x \); that is, \( E_L(x) \) is the Fitting null-component of \( L \) relative to \( \text{ad} \ x \).

First we consider the kernel of a nilpotent derivation of \( L \).

**Lemma 1.1.** Let \( D \) be a nonzero nilpotent derivation of a Lie algebra \( L \) over an arbitrary field. Assume that \( x \in \text{Ker} D \) acts nilpotently on \( \text{Ker} D \). Then \( x \) acts nilpotently on \( L \).

**Proof.** Suppose that \( \text{ad} \ x \) is not nilpotent on \( L \). Let \( L = E_L(x) \oplus L_1(x) \) be the Fitting decomposition of \( L \) relative to \( \text{ad} \ x \). We have \( L_1(x) \neq 0 \). Since \( x \in \text{Ker} D \) we find

\[
[\text{ad} \ x, D] = \text{ad} \ D(x) = 0
\]

so that \( D \) and \( \text{ad} \ x \) commute. Then \( L_1(x) \) is stabilized by \( D \). Since \( D \) is nilpotent, we have that \( \text{Ker} D \cap L_1(x) \neq 0 \). On the other hand, since \( x \) acts nilpotently on \( \text{Ker} D \) it follows that \( \text{Ker} D \leq E_L(x) \). We find, \( 0 \neq \text{Ker} D \cap L_1(x) \leq E_L(x) \cap L_1(x) = 0 \), a contradiction.

**Theorem 1.2.** Let \( F \) be an arbitrary perfect field. Then the following statements are equivalent:

1. Any Lie algebra over \( F \) contains ad-nilpotent elements.
2. There are no simple Lie algebras over \( F \) having only abelian subalgebras.

**Proof.** To prove (1) implies (2). Suppose that \( L \) is a simple Lie algebra over \( F \) having only abelian subalgebras. By using Theorem 4.1 of [6] we obtain that \( \text{ad} \ x \) is semisimple for every \( x \in L \). By (1), there exists an element \( y \in L \) such that \( \text{ad} y \) is nilpotent on \( L \). This yields \( y \in Z(L) = 0 \), a contradiction.

In order to prove the converse, suppose that \( L \) is a Lie algebra over \( F \) which contains no nonzero ad-nilpotent elements and \( L \) is of minimal dimension. First we claim that \( \text{ad} \ x \) is semisimple for every \( x \in L \). To prove this, let \( D \) be a nonzero nilpotent derivation of \( L \). Then \( \text{Ker} D \) is a proper subalgebra of \( L \). By the minimality of \( \dim L \), we have that \( \text{Ker} D \) contains a nonzero element \( z \) such that
ad $z$ is nilpotent in $\text{Ker} \, D$. From Lemma 1.1 it follows that $ad \, z$ is nilpotent on $L$, a contradiction. Therefore, $L$ has no nonzero nilpotent derivation. The claim now follows from the Jordan-Chevalley decomposition (see [7, Chapter 2, Theorem 3.5 and Proposition 3.3]).

Now let $S$ be a proper subalgebra of $L$. Assume $Z(S) \neq S$. Then as $\dim S/Z(S) < \dim L$, we have that there exists $u \in S - Z(S)$ such that $u$ acts nilpotently on $S/Z(S)$. We see that $u$ acts nilpotently on $S$. Since $ad \, z$ is semisimple by the claim, we conclude that $u \in Z(S)$ which is a contradiction. Therefore, $S$ is abelian.

Finally, we show that $L$ is simple. Suppose that $N$ is a proper ideal of $L$. Let $x \in N - (0)$. Then $[x, L] \leq N$ whence $[x[x, L]] = 0$ since $N$ is abelian. This yields $ad \, x$ is nilpotent on $L$, a contradiction. This completes the proof.

A field $F$ is said to be a field of type $(C_1)$ if every homogeneous polynomial $f(X_1, \ldots, X_n)$ over $F$ of degree less than the number $n$ of variables has a nontrivial root in $F^n$.

We recall that every finite field is of type $(C_1)$ by Chevalley's Theorem (see [10]).

**Corollary 1.3.** Let $F$ be a perfect field of type $(C_1)$ of arbitrary characteristic. Then every Lie algebra over $F$ contains ad-nilpotent elements.

**Proof.** Suppose that the assertion is not true. Then, by Theorem 1.2, there exists a simple Lie algebra $L$ over $F$ having only abelian subalgebras. Let $x \in L - (0)$. By [6], $ad \, x$ is semisimple and so $E_L(x)$ is a proper subalgebra of $L$. Since $E_L(x)$ is self-normalizing by [2], it follows that $E_L(x)$ is a Cartan subalgebra of $L$.

Now suppose $\text{char}(F) = 0$. Then, from the classical theory it follows that all Cartan subalgebras of $L$ have the same dimension. On the other hand, we consider the characteristic polynomial $f(\lambda)$ of $L$ relative to a basis $(e_i)$ of $L$ (see [9]). Write
\[
(*) \quad f(\lambda) = \lambda^n - r_1(X_1, \ldots, X_n)\lambda^{n-1} + \cdots + (-1)^{n-1}r_{n-1}(X_1, \ldots, X_n)\lambda + r_n
\]
where $r_i(X_1, \ldots, X_n)$ is a homogeneous polynomial of degree $i$. Take an element $x \in L - (0)$ and decompose $x = t_1e_1 + \cdots + t_ne_n$ with $t_i \in F$. Then, the characteristic polynomial of $ad \, x$ is obtained by specializing $X_i = t_i$, $i = 1, \ldots, n$ in $(*)$. Pick $r$ such that $r_{n-r}(t_1, \ldots, t_n) \neq 0$ and $r_{n-i}(t_1, \ldots, t_n) = 0$ for $i > r$. We have $\dim E_L(x) = r$. Now consider the polynomial $r_{n-r}(X_1, \ldots, X_n)$. As $F$ is of type $(C_1)$, we can take a nontrivial root $(s_1, \ldots, s_n) \in F^n$ of $r_{n-r}$. Write $y = s_1e_1 + \cdots + s_ne_n$. We find $\dim E_L(y) \neq r$ so $\dim E_L(x) \neq \dim E_L(y)$, which is a contradiction.

Suppose then $\text{char}(F) = p > 0$. Now we consider the $p$-closure $C$ of $L$ in $\text{Der}(L)$, where $\text{Der}(L)$ denotes the derivation algebra of $L$. The proof of Theorem 7.2 of [6] shows that $C$ is $p$-simple and every proper $p$-subalgebra of $C$ is abelian. Assume that $u$ is a nonzero nilpotent element of $C$. Since $L$ is an ideal of $C$, we have that $u$ acts nilpotently on $L$. As $L$ has no nonzero nilpotent derivation by Theorem 4.1 of [6], we find $[uL] = 0$. But clearly the centralizer $C_C(L)$ of $L$ in $C$ is a $p$-ideal of $L$, so $C_C(L) = 0$ whence $u = 0$, which is a contradiction. We conclude that $C$ contains no nonzero nilpotent elements. Then, since $F$ is perfect, we have that every element of the restricted Lie algebra $C$ is semisimple.

Now, let $x \in C - (0)$ and consider the Engel subalgebra $E_C(x)$. We have that $E_C(x)$ is a $p$-subalgebra since it is self-normalizing. Then, by above, $E_C(x)$ is a toral Cartan subalgebra of $C$. From [11] it follows that all Engel subalgebras of $C$ have the same dimension. On the other hand, as in the characteristic zero case, we can
find two elements $x, y \in C$ such that $\dim E_C(x) \neq \dim E_C(y)$. This contradiction completes the proof.

**Corollary 1.4.** Let $F$ be a perfect field over which every Lie algebra contains an ad-nilpotent element. Then the Brauer group $\text{Br}(F)$ of $F$ is trivial.

**Proof.** This result follows from Theorem 1.2 and Theorem 8.5 of [6].

Next, by using Lemma 1.1, we obtain a sufficient condition to insure the existence of an ad-nilpotent element in a suitable subset of a Lie algebra over a perfect field of arbitrary characteristic.

**Theorem 1.5.** Let $L$ be a Lie algebra over a perfect field $F$ of arbitrary characteristic. Assume that every subalgebra $T$ of $L$ contains at least one element $x \neq 0$ with $\text{ad} x$ split on $T$. Let $V$ be a vector subspace of $L$ consisting of ad-split elements. Assume for every $y \in L - V$ that $V$ contains no nonzero subspace invariant under $\text{ad} y$. Then every subalgebra $S$ of $L$ not contained in $V$ has a nonzero element $z \in S - V$ such that $\text{ad} z$ is nilpotent on $S$.

**Proof.** Let $S$ be a subalgebra of $L$ such that $S \notin V$. We argue by induction on $\dim S$. For $\dim S = 1$ it is trivial. Then suppose $\dim S = r > 1$ and the result holds for subalgebras of dimension less than $r$. Let $x \in S - V$ and let $\text{ad}_S x = D_n + D_s$ be the Jordan-Chevalley decomposition of $\text{ad}_S x$. Assume $D_n \neq 0$. Then $\text{Ker} D_n$ is a proper subalgebra of $S$. We have,

$$\text{ad}_S(D_n(x)) = [\text{ad}_S x, D_n] = [D_s, D_n] = 0$$

whence $D_n(x) \in Z(S)$. But $Z(S) \cap V = 0$ by our hypothesis on $V$. This yields $D_n(x) = 0$, otherwise we would have $D_n(x) \notin V$ with $D_n(x)$ acting nilpotently on $S$. Therefore, $x \in \text{Ker} D_n$ so that $\text{Ker} D_n \notin V$. Now, by the inductive hypothesis, there exists $y \in \text{Ker} D_n - V$ such that $\text{ad} y$ is nilpotent on $\text{Ker} D_n$. But then from Lemma 1.1 it follows that $\text{ad} y$ is nilpotent on $S$. We may assume then that $\text{ad} z$ is semisimple for every $z \in S - V$.

Next, take $z \in S$ such that $\text{ad} z$ is split on $S$. Assume $z \notin V$. Let $\alpha$ be an eigenvalue of $\text{ad} z$. We have $[xz] = \alpha x$ for some $x \in S - 0$. We find $[xz[xz]] = 0$ so that $(\text{ad}_S x)^2(z) = 0$. But since $z \notin V$ it follows that $z \notin V$ by our hypothesis on $V$ again. Then, by above, $\text{ad}_S x$ is semisimple and hence $(\text{ad} x)(z) = 0$. This yields $\alpha = 0$, thus $\text{ad} z$ is nilpotent on $S$. We may suppose then that $z \in V$. In particular we have that $S \cap V \neq 0$ and then $Z(S) = 0$ by our hypothesis on $V$.

Now let $T$ be a subalgebra of $L$ maximal with respect to be contained in $S \cap V$. By above, $T \neq 0$. If every element of $T$ acts nilpotently on $S$, then from Engel's Theorem it follows that there exists $u \in S - T$ such that $[u, T] \leq T$. But then, since $T$ is invariant under $\text{ad} u$, we find that $u \in V$ by our hypothesis on $V$. This yields that the subalgebra $T + Fu$ is contained in $S \cap V$, which contradicts the maximality of $T$. We conclude that there exists an element $v \in V \cap S$ such that $\text{ad} v$ is not nilpotent on $S$.

Finally, let $D$ be the semisimple component of $\text{ad}_S v$. As $v \in V$, we have that $\text{ad}_S v$ is split. Thus $D$ is split too, and $D \neq 0$ since $\text{ad}_S v$ is not nilpotent. Let $S = S_0(D) \oplus S_0(D) \oplus \cdots \oplus S_r(D)$ be the decomposition of $S$ into the eigenspace of $D$. We have $S_0(D) = \text{Ker} D$. Suppose $\text{Ker} D \notin V$. Then, since $\text{Ker} D$ is a proper subalgebra of $S$ there exists $x \in (\text{Ker} D) - V$ such that $\text{ad} z$ is nilpotent on $\text{Ker} D$. **
by the inductive hypothesis. Since $x \notin V$, we have that $\text{ad} \, x$ is semisimple on $S$. It follows that $x$ acts trivially on $S$. Thus $(\text{Ker} \, D) \cap V = 0$ by our hypothesis on $V$. But, $\text{ad}_S(D(v)) = [\text{ad}_S \, v, D] = 0$, whence $D(v) \in Z(S) = 0$, so that $v \in (\text{Ker} \, D) \cap V$, which is a contradiction. We conclude that $S_0(D) \leq V$. This yields $S_\alpha(D) \notin V$ for some $\alpha \neq 0$ since $S \notin V$ and $V$ is a vector subspace of $L$. Pick $e \in S_\alpha(D) - V$. We have $D(e) = \alpha e$. And then $[\text{ad}_S \, e, D] = \text{ad}_S(D(e)) = \alpha(\text{ad}_S \, e)$, whence $[\text{ad}_S \, e, \text{ad}_S(D(e))] = 0$. On the other hand, since $e \in S - V$ we find that $\text{ad}_S \, e$ is semisimple, and thus $\text{ad}(\text{ad}_S \, e) \colon \text{Der}(S) \rightarrow \text{Der}(S)$ is semisimple. This yields $[\text{ad}_S \, e, D] = \text{ad}_S(D(e)) = 0$ so $D(e) = 0$ and hence $\alpha = 0$. This contradiction completes the proof.

**Corollary 1.6.** Let $L$ be a Lie algebra over a perfect field of arbitrary characteristic. Assume that every subalgebra $S$ of $L$ contains at least one element $x$ such that $\text{ad} \, x$ is split on $S$. Then every subalgebra $T$ of $L$ contains an element $y$ with $\text{ad} \, y$ nilpotent on $T$.

**Proof.** It follows from Theorem 1.5 by taking $V$ to be zero.

**2. Lie algebras with subalgebras of codimension one.** In this section we consider a situation in which Theorem 1.5 applies. By using this Theorem, we quickly prove that a Lie algebra $L$ with a subalgebra $M$ of codimension 1 which contains no nonzero ideals of $L$ necessarily contains a self-centralizing ad-nilpotent element, whenever the ground field $F$ is perfect of arbitrary characteristic. From this result and Theorem 3.2 of [4] follows an easy proof of Amayo's assertion, whenever $F$ is perfect with $\text{char}(F) \neq 2$.

The proof of the following Lemma is reproduced from the proof of Lemma 2.1 of [1]. We include it here for completeness.

**Lemma 2.1.** Let $L$ be a finite dimensional Lie algebra over an arbitrary field $F$. Suppose $M \leq L$ is a subalgebra of codimension one which contains no nonzero ideals of $L$. Then $M$ is supersolvable.

**Proof.** Define $M_0 = M$, $M_{i+1} = \{x \in M_i|[x, L] \leq M_i\}$ for $i \geq 0$. We have $[M_i, M_j] \leq M_{i+j}$, in particular $M_i \leq M$ for all $i$. If $M_i = M_{i+1}$, then $M_{i+1}$ is an ideal of $L$, thus $M_{i+1}$ must be zero and hence $M_i = 0$. Then we have $M = M_0 > M_1 > \cdots > M_r = 0$ for some $r$.

Now let $a \in L - M$. We claim that $M_{i+1} = \{x \in M_i|(\text{ad} \, a)^{i+1}(x) \in M\}$ for $i \geq 0$. It is clear for $i = 0$ since $L = Fa$. Suppose that $i \geq 1$ and the result holds for $i$. If $x \in M_i$ with $(\text{ad} \, a)^{i+1}(x) \in M$, then $(\text{ad} \, a)^i[xa] \in M$ and $[xa] \in M_{i-1}$, whence $[xa] \in M_i$. We have $[x, L] \leq [x, M] + [x, Fa] \leq [M_i, M] + M_i = M_i$, so that $x \in M_{i+1}$. The claim follows.

Next, for $i \geq 0$ we define the linear map $\sigma_i : M \rightarrow F$ by means of $(\text{ad} \, a)^{i+1}(x) \equiv \sigma_i(x)a \pmod{M}$. We have $\text{Ker} \, \sigma = M_{i+1}$ by above. Therefore, $\dim M_i/M_{i+1} = 1$ for $i = 0, 1, \ldots, r - 1$. Thus $M$ is supersolvable.

**Theorem 2.2.** Let $L$ be a Lie algebra over a perfect field $F$ of arbitrary characteristic. Suppose $M \leq L$ is a subalgebra of codimension one which contains no nonzero ideals of $L$. Then $L$ contains a self-centralizing ad-nilpotent element.

**Proof.** By Lemma 2.1, $M$ is supersolvable so that $\text{ad} \, y$ is split on $M$ for every $y \in M$. As $\dim L/M = 1$, we have that $\text{ad} \, y$ is split on $L$ for every $y \in M$. On
the other hand, the proof of Lemma 3.7 of [4] shows for every \( x \in L - M \) that \( M \)
contains no nonzero subspace invariant under \( \text{ad} \, x \).

Now, let \( S \) be a subalgebra of \( L \) such that \( S \not\subset M \). If \( \dim S = 1 \), write \( S = Fx \),
then obviously, \( \text{ad} \, x \) is split on \( S \). Now suppose \( \dim S > 1 \). Since \( \dim L/M = 1 \), we
have \( S \cap M \neq 0 \). Then \( S \) contains a nonzero element \( y \) such that \( \text{ad} \, y \) is split on \( S \).

Therefore, Theorem 1.5 applies and there exists \( x \in L - M \) such that \( \text{ad} \, x \) is
nilpotent.

Finally, by Lemma 3.8 of [4] we have that \( C_L(x) = Fx \). The proof is complete.

**Corollary 2.3.** Let \( L \) be a Lie algebra over a perfect field \( F \) with \( \text{char}(F) \neq 2 \).
Assume that \( L \) has a subalgebra of codimension 1 which contains no nonzero ideals
of \( L \). Then, \( L \) is either 2-dimensional, isomorphic to \( \text{sl}(2, F) \) or a Zassenhaus
algebra.

**Proof.** It follows from Theorem 2.2 and Theorem 3.2 of [4].

**References**

   Math. 577, 17006.
7. R. Farnsteiner and H. Strade, Modular Lie algebras and their representations, Marcel Dekker,
   636–643.
11. A. A. Premet, Toroidal Cartan subalgebras of p-algebras, and anisotropic Lie algebras of positive
    Zbl. Math. 597, 17007.

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